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On graphs with α- and b-edge consecutive edge magic labelings

Christian Barrientos

Valencia College, Orlando, FL 32832, U.S.A.

chr_barrientos@yahoo.com

Abstract

Among the most studied graph labelings we have the varieties called alpha and edge-magic. Even when their definitions seem completely different, these labelings are related. A graceful labeling of a bipartite graph is called an α-labeling if the smaller labels are assigned to vertices of the same stable set. An edge-magic labeling of a graph of size n is said to be b-edge consecutive when its edges are labeled with the integers $b+1, b+2, \ldots, b+n$, for some $0 \leq b \leq n$. In this work, we prove the existence of several b-edge consecutive edge-magic labelings for any graph of order m and size $m-1$ that admits an α-labeling. In addition, we determine the exact value of b induced by the α-labeling, as well as for its reverse, complementary, and reverse complementary labelings.

Keywords: α-labeling, edge-magic graph, b-edge consecutive
Mathematics Subject Classification : 05C78

1. Introduction

Let G be a graph of order m and size n; the graph G is said to be *edge-magic* if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, m+n\}$ such that $f(u) + f(v) + f(uv) = k$ for all $uv \in E(G)$, where k is a constant. We refer to k as the *valence* of f; some authors called k the *magic constant* of f. This type of total labeling was originally introduced by Kotzig and Rosa [8]. Enomoto et al. [5] said that an edge-magic labeling is *super* when the set of vertex labels is $\{1, 2, \ldots, m\}$. The proof of the following lemma can be found in [6].

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Lemma 1.1. *A graph G of order m and size n is super edge-magic if and only if there exists a bijection $f : V(G) \rightarrow \{1, 2, \dots, m\}$ such that the set $\{f(u) + f(v) : uv \in E(G)\}$ consists of n consecutive integers.*

An outline of the proof of this result is the following. Assuming that for a graph G of order m and size n, there exists a bijection $f : V(G) \rightarrow \{1, 2, \dots, m\}$ such that $\{f(u)+f(v) : uv \in E(G)\}$ is formed by n consecutive integers, then f can be extended to a super edge-magic labeling by defining $f(uv) = m + n - i$, where $i = f(u) + f(v) - \min\{f(x) + f(y) : xy \in E(G)\}$. Note that the valence of f is $m + n + \min\{f(x) + f(y) : xy \in E(G)\}$. In Figure 1 we show an example of this labeling for the graph $2C_8 \cup P_2$; in this example $m = 18$, $n = 17$, and the valence of the labeling is $k = 46$.

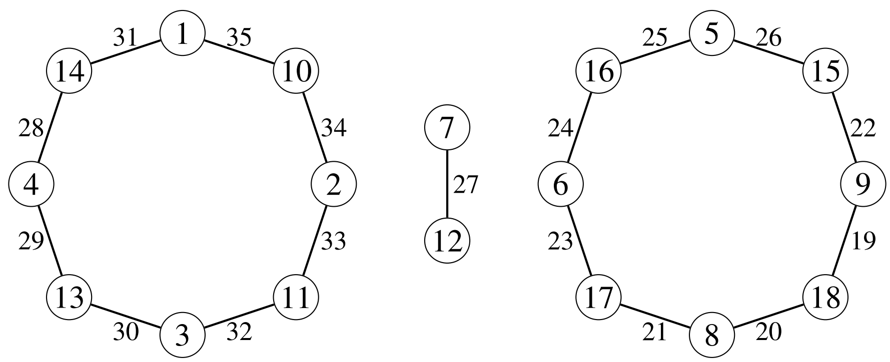

Figure 1. Super edge-magic labeling of $2C_8 \cup P_2$ where the valence is $k = 46$

Sugeng and Miller [12] said that an edge-magic labeling is *b-edge consecutive* when the set of edge labels is $\{b + 1, b + 2, \dots, b + n\}$ for some $b \in \{0, 1, \dots, n\}$. Based on this definition we can see that any super edge-magic labeling of a graph of order m is an m-edge consecutive edge-magic labeling. Let f be a b-edge consecutive edge-magic labeling of a graph of order m and size n. The *dual* labeling of f (also called the *complementary* of f) is the labeling $\overline{f}$ defined as $\overline{f}(x) = m + n + 1 - f(x)$ for every $x \in V(G) \cup E(G)$. It is well-known that $\overline{f}$ is an edge-magic labeling; moreover, $\overline{f}$ is indeed a b-edge consecutive labeling because f is b-edge consecutive; therefore, $\{m + n + 1 - f(x) : x \in E(G)\}$ is a set of n consecutive integers. We must observe that if k is the valence of f, then the valence of $\overline{f}$ is $3(m + n + 1) - k$.

Two results about b-edge consecutive edge-magic labelings, proven in [12], that are relevant in this work are:

Theorem 1.1. *Let G be a connected graph of order m. If there is a b-edge consecutive edge-magic labeling of G, for some $b \in \{1, 2, \dots, m - 1\}$, then G is a tree.*

Theorem 1.2. *If G is a caterpillar of size n, then G admits a b-edge consecutive edge-magic labeling, where the value of b depends on the diameter and the number of leaves.*

In [13], Sugeng and Silavan extended the results in [12], by providing several classes of trees that admit this type of total labeling. They showed that regular caterpillars, firecrackers, caterpillar-like trees, path-like trees, and banana trees, are b-edge consecutive edge-magic graphs; we must note that the word regular has a different connotation for each variety of tree considered in [13].

Let G be a bipartite graph of order m and size n which stable sets are S_1 and S_2. Rosa[1] [10] defined an *α-labeling* of G as an injective function $f : V(G) \to \{1, 2, \ldots, n+1\}$ such that $\{|f(u) - f(v)| : uv \in E(G)\} = \{1, 2, \ldots, n\}$ and for every $(u, v) \in S_1 \times S_2$, $f(u) < f(v)$. The *boundary value* of f is $\max\{f(u) : u \in S_1\}$. As an immediate application of this kind of labeling, Rosa proved that there exists a cyclic decomposition of K_{2tn+1} into copies of any graph of size n that admits an α-labeling. An *α-graph* is any graph that can be α-labeled. Among other results, Rosa showed that all caterpillars are α-trees. Several papers have followed Rosa' seminal work and multiple classes of α-trees are known. For example, in [9], Kotzig proved that almost all trees can be α-labeled; other two families of α-trees are the path-like trees [1] and the triangular trees [4]. Gallian [7] devotes an entire section of his survey to this labeling.

Suppose that G is an α-graph. Let S_1 and S_2 be the stable sets of G, where $s_1 = |S_1|$ and $s_2 = |S_2|$. Assuming that f is an α-labeling of G that assigns the label 1 to a vertex of S_1, then the valence of the m-edge consecutive edge-magic labeling obtained from f is $k = 2m + 1 + s_1$. The *complementary labeling* of f, which is defined by $\overline{f}(v) = m + 1 - f(v)$ for each $v \in V(G)$, places the label 1 on an element of S_2. Since $\overline{f}$ is also an α-labeling of G, the m-edge consecutive edge-magic labeling obtained using $\overline{f}$ instead of f, has valence $k = 2m + 1 + s_2$.

Several years after the publication of his seminal paper on difference vertex labeling, Rosa [11] introduced the following definition. Let f be an α-labeling of a graph G of size n that assigns the label 1 to a vertex of S_1; the α-labeling f_r of G given by

$$f_r(v) = s_1 + 1 - f(v) \pmod{n+1}$$

is called *inverse labeling* of f. This labeling is also called *reverse labeling* of f. In general, $f \neq f_r$ but they have the same boundary value. Consequently, if f is an α-labeling of G, then $\overline{f}$, f_r, and $\overline{f}_r$ are also α-labelings of G, where f and f_r have boundary value s_1, while $\overline{f}$ and $\overline{f}_r$ have boundary value s_2. With all these facts, the proof of the following result is straightforward.

Theorem 1.3. *Let G be an α-graph of order m and size $m-1$, and let f be an α-labeling of G such that the vertex labeled 1 by f is in S_1. If $f \neq f_r$, then there are four m-edge consecutive edge-magic labelings of G, two with valence $2m+1+s_1$ and two with valence $2m+1+s_2$.*

In Figure 2 we show the labelings f, f_r, $\overline{f}$, and $\overline{f}_r$ for a tree of order 11 with stable sets of cardinalities 5 and 6; in addition, we show the corresponding 11-edge consecutive edge-magic labelings.

In this work, we continue the study of b-edge consecutive edge-magic labelings initiated by Sugeng and Miller, by proving that if G is an α-graph of order m and size $m-1$, then G also admits a b-edge consecutive edge-magic labeling, where b is any of the elements in $\{0, m, |S_1|, |S_2|\}$.

[1] In the original definition, the codomain of f is $\{0, 1, \ldots, n\}$.

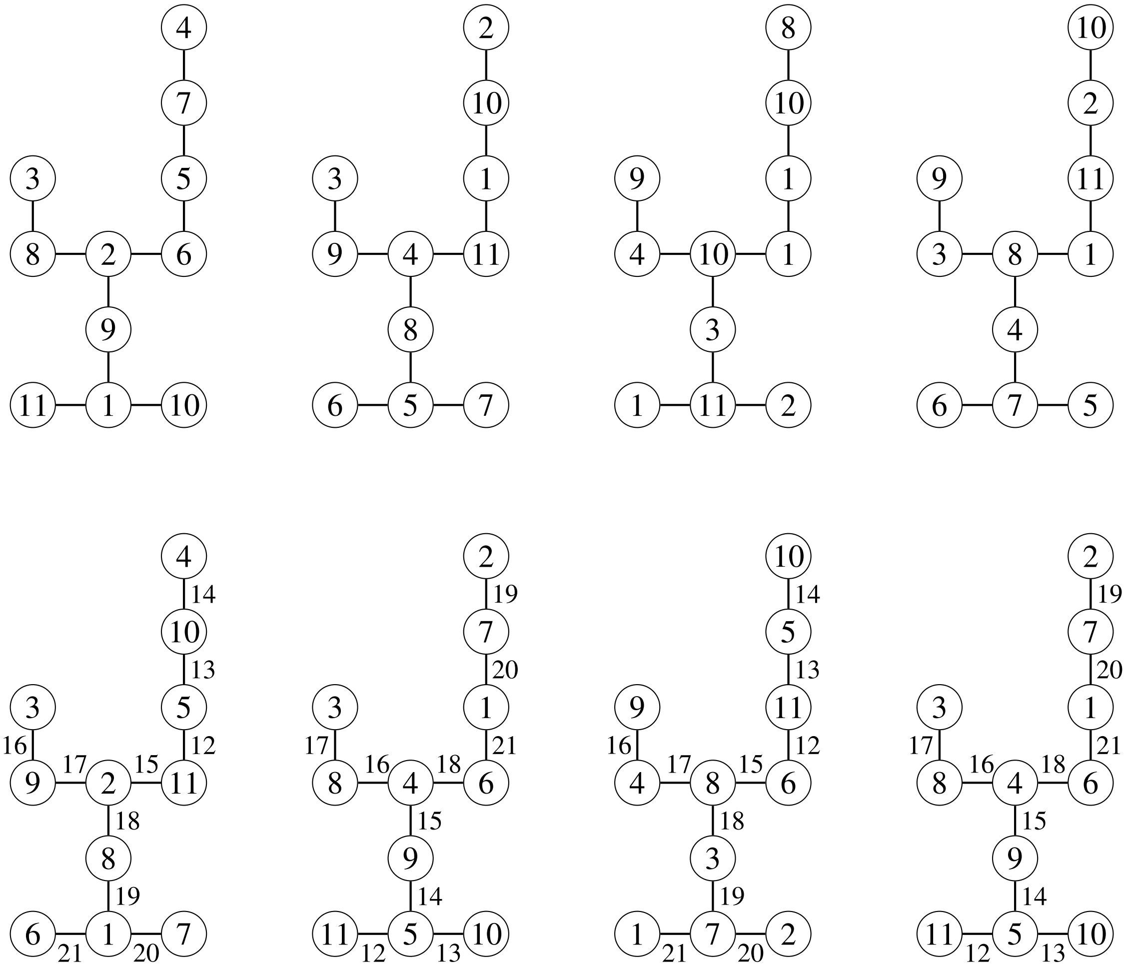

Figure 2. The α-labelings f, f_r, $\overline{f}$, and $\overline{f}_r$ of a tree of order 11 together with their associated b-edge consecutive edge-magic labelings

2. Main Results

In the introduction we mentioned that a super edge-magic labeling of a graph of size m is an m-edge consecutive edge-magic labeling. Figueroa-Centeno et al. [6] proved that if G is an α-graph of order m and size $m-1$, then G is super edge-magic. Formulating this result in terms of b-edge consecutive edge-magic labelings we have the following.

Theorem 2.1. *If G is an α-graph of order m and size $m-1$, then G admits an m-edge consecutive edge-magic labeling.*

Theorem 2.2. *If G is an α-graph of order m and size $m-1$, then G admits a 0-edge consecutive edge-magic labeling.*

Proof. Suppose that f is an α-labeling of G. By Theorem 2.1 we know that f can be transformed into a m-edge consecutive edge-magic labeling of G. Let g denote the m-edge consecutive labeling of G obtained using f. Then, the set of edge labels is $\{m+1, m+2, \ldots, 2m-1\}$. Therefore, the set of labels assigned by $\overline{g}$ to the edges of G is $\{1, 2, \ldots, m-1\}$. Consequently, $\overline{g}$ is a 0-edge consecutive edge-magic labeling of G. □

In the next result we prove that for any α-graph of order m and size $m-1$, with stable sets of cardinalities s_1 and s_2, there exists a b-edge consecutive edge-magic labeling, where b is either s_1 or s_2.

Theorem 2.3. *If G is an α-graph of order m and size $m-1$, then there exists a b-edge consecutive edge-magic labeling of G where b is the cardinality of any of its stable sets.*

Proof. Suppose that G is an α-graph of order m and size $m-1$, with stable sets S_1 and S_2, where $|S_1| = s_1$ and $|S_2| = s_2$. Since G is an α-graph, there is an α-labeling f of G that assigns the label 1 to a vertex of S_1. Consider the following labeling of the vertices of G:

$$g(v) = \begin{cases} f(v) & \text{if } v \in S_1, \\ 2m + s_1 - f(v) & \text{if } v \in S_2. \end{cases}$$

Thus, the labels assigned by g to the elements of S_1 form the set $\{1, 2, \ldots, s_1\}$, while the elements of S_2 receive the labels in $\{m+s_1, m+s_1+1, \ldots, 2m-1\}$. Let $uv \in E(G)$ such that $f(v)-f(u) = w$ for some $w \in \{1, 2, \ldots, m-1\}$; then

$$\begin{aligned} g(v) + g(u) &= 2m + s_1 - f(v) + f(u) \\ &= 2m + s_1 - \big(f(v) - f(u)\big) \\ &= 2m + s_1 - w. \end{aligned}$$

Since $1 \leq w \leq m-1$, we get that $m + s_1 + 1 \leq g(u) + g(v) \leq 2m + s_1 - 1$. In other terms, $\{g(u) + g(v) : uv \in E(G)\}$ is a set of $m-1$ consecutive integers.

We extend the labeling g to include the edges of G. Let $uv \in E(G)$ such that $g(u) + g(v) = 2m + s_1 - i$ for some $i \in \{1, 2, \ldots, m-1\}$, then $g(uv) = m + s_1 + i$. Hence, the labels assigned to the edges of G form the set $\{s_1 + 1, s_1 + 2, \ldots, m + s_1 - 1\}$. Consequently, the labels assigned by g to the edges of G are $m-1$ consecutive integers; in addition, the labels on $V(G) \cup E(G)$ form the set $\{1, 2, \ldots, 2m-1\}$. Therefore, g is a s_1-edge consecutive edge-magic labeling. Since $\{g(u) + g(v) : uv \in E(G)\} = \{m + s_1 + i : 1 \leq i \leq m-1\}$, we get that when uv is the edge of G such that $g(u) + g(v) = m + s_1 + i$, then

$$g(u) + g(v) + g(uv) = m + s_1 + i + m + s_1 - i = 2(m + s_1).$$

Thus, the valence of g is $k = 2(m + s_1)$. Then, g is a s_1-edge consecutive edge-magic labeling of G with valence $k = 2(m + s_1)$.

If we use $\overline{f}$ instead of f, the resulting labeling is s_2-edge consecutive edge-magic, and its valence is $k = 2(m + s_2)$.

□

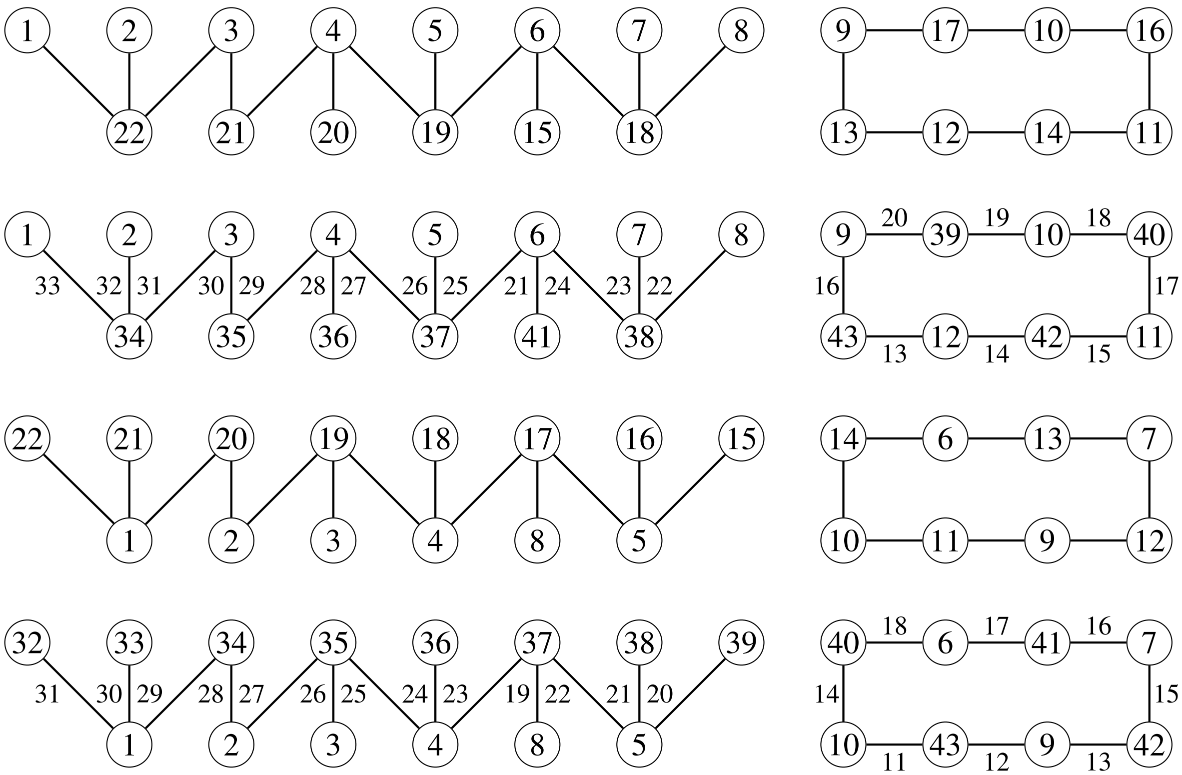

Figure 3. The α-labelings f and $\overline{f}$ of a disconnected graph of order 22 and size 21 together with their associated b-edge consecutive edge-magic labelings

In Figure 3 we show the two b-edge consecutive edge-magic labelings obtained using f and $\overline{f}$ for a disconnected graph of order 22, size 21, with stable sets of cardinalities 10 and 12.

Since any tree of order m has size $m-1$, we deduce that any α-tree admits a b-edge consecutive edge- magic labeling where b is the cardinality of any of its stable sets.

Corollary 2.1. *Let T be an α-tree with a stable set of cardinality b, then T admits a b-edge consecutive edge-magic labeling.*

Some families of disconnected α-graphs, of order m and size $m-1$, are known. Let x and y be positive integers such that $y \geq x+2$; in [2], Barrientos and Minion proved that if G_y is a caterpillar of size y with stable sets S_1 and S_2, such that there exist $v \in S_1$ adjacent to a leaf, and an α-labeling f of G_y with the property that $f(v) = s_1 - x - 1$, then $C_{4x} \cup G_y$ is an α-graph. The graph depicted in Figure 3 satisfies the conditions described above, where the vertex v is the vertex labeled 6 in the first representation. Barrientos and Minion [3] continued their work about disconnected graphs that admit an α-labeling; they proved that if G is an α-graph of order and size y, then $tG \cup P_t$ is an α-graph for every positive integer t, where P_t is the path of order t. Let L_{t-1} be any linear forest of size $t-1$; Barrientos and Minion [3] proved that if G is an α-graph of order m and size n, with $m < n$, then $tG \cup L_{t-1}$ is an α-graph for every $t \geq 2$. Therefore, all these graphs admit a b-edge consecutive edge-magic labeling for some values of b.

3. Conclusions

Suppose that G is a super edge-magic graph of size m. We have presented the fact that a super edge-magic labeling of G is an m-edge consecutive labeling with valence k. We proved that the existence of this m-edge consecutive labeling implies the existence of a 0-edge consecutive edge-magic labeling as well; the valence of this new labeling is $6m - k$. In the case where G is a bipartite graph, with stable sets S_1 and S_2, where $|S_1| = s_1$ and $|S_2| = s_2$, an m-edge consecutive edge-magic labeling exists, provided that G is an α-graph; the valence of the m-edge consecutive labeling is $k = 2m+1+s_1$ or $k = 2m+1+s_2$, depending on whether the vertex labeled 1 belongs to S_1 or S_2, respectively.

We know now that when G is an α-graph, it also admits a b-edge consecutive edge-magic labeling, where $b = s_1$ or $b = s_2$, whose valences are $k = 2m + 1 + s_1$ or $k = 2m + 1 + s_2$, respectively. Furthermore, if we use the reverse labeling of the α-labeling f, we obtain a different b-edge consecutive edge-magic labeling, where the parameters b and k are the same for the labelings f and f_r.

Since in the definition of a b-edge consecutive edge-magic labeling of a graph of size n, we have that b could be any element of $\{0, 1, \ldots, n\}$, we may ask for which values of b such a labeling exists when the graph is an α-graph.

References

[1] C. Barrientos, *Difference Vertex Labelings*, Ph.D. Thesis, Universitat Politècnica de Catalunya, Barcelona, 2004.

[2] C. Barrientos and S. Minion, Constructing graceful graphs with caterpillars, *J. Algor. Computation*, **48** (2016), 117–125.

[3] C. Barrientos and S. Minion, New α-trees and graceful unions of α-graphs and linear forests, *J. Combin. Math. Combin. Comput.*, **108** (2019), 205–220.

[4] C. Barrientos, Alpha graphs with different pendent paths, *Elect. J. Graph Th. Appl.*, **8** (2) (2020), 301–317.

[5] H. Enomoto, A. S. Llado, T. Nakamigawa, and G. Ringel, Super edge-magic graphs, *SUT J. Math.*, **34** (1998) 105–109.

[6] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.*, **231** (2001) 153–168.

[7] J. A. Gallian, A dynamic survey of graph labeling. *Electronic J. Combin.*, **24**(#DS6.), 2021.

[8] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), 451–461.

[9] A. Kotzig, On certain vertex valuations of finite graphs, *Util. Math.*, **4** (1973), 67–73.

[10] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966)*, Gordon and Breach, N. Y. and Dunod Paris (1967) 349–355.

[11] A. Rosa, Labelling snakes, *Ars Combin.*, **3** (1977) 67–74.

[12] K. A. Sugeng and M. Miller, On consecutive edge magic total labelings of graphs, *J. Discrete Algorithms*, **6** (2008) 59–65.

[13] K. A. Sugeng and D. R. Silaban, On b-edge consecutive edge labeling of some regular trees, *Indonesian J. Combin.*, **4** (1) (2020), 76–81.