Indonesian Journal of Combinatorics **4** (2) (2020), 114–125

INDONESIAN JOURNAL OF COMBINATORICS

The forcing monophonic and the forcing geodetic numbers of a graph

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Abstract

For a connected graph $G = (V, E)$, let a set S be a m-set of G. A subset $T \subseteq S$ is called a forcing subset for *S* if *S* is the unique m-set containing *T*. A forcing subset for *S* of minimum cardinality is a minimum forcing subset of *S*. The forcing monophonic number of *S*, denoted by $f_m(S)$, is the cardinality of a minimum forcing subset of *S*. The forcing monophonic number of *G*, denoted by $f_m(G)$, is $f_m(G) = min\{f_m(S)\}$, where the minimum is taken over all minimum monophonic sets in *G*. We know that $m(G) \leq g(G)$, where $m(G)$ and $g(G)$ are monophonic number and geodetic number of a connected graph *G* respectively. However there is no relationship between $f_m(G)$ and $f_g(G)$, where $f_g(G)$ is the forcing geodetic number of a connected graph *G*. We give a series of realization results for various possibilities of these four parameters.

Keywords: geodetic number, monophonic number, forcing geodetic number, forcing monophonic number
Mathematics Subject Classification : 05C12, 05C38
DOI: 10.19184/ijc.2020.4.2.5

1. Introduction

By a *graph G* = (*V*,*E*), we mean a finite undirected connected graph without loops or multiple edges. The *order and size* of *G* are denoted by *p* and *q* respectively. For basic graph theoretic terminology, we refer to Harary [1]. The *distance* $d(u, v)$ between two vertices *u* and *v* in a connected graph *G* is the length of shortest $u - v$ path in *G*. An $u - v$ path of length $d(u, v)$ is called an $u - v$

Received: 9 February 2020, Revised: 13 December 2020, Accepted: 26 December 2020.

geodesic. A vertex x is said to be lie a $u-v$ geodesic P if x is a vertex of P including the vertices u and v. A *geodetic set* of G is a set $S \subseteq V$ such that every vertex of G is contained in geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a *minimum geodetic set* or simply a g-set of G. The geodetic number of a graph was introduced in [1] and further studied in [3, 4, 5, 7, 8, 9, 16, 17, 18, 20, 23, 25]. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique g-set of G containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing geodetic number* of S, denoted by $f_g(S)$, is the cardinality of a minimum forcing subset of S. The *forcing geodetic number* of G, denoted by $f_g(G)$, is $f_g(G) = min\{f_g(S)\}$, where the minimum is taken over all minimum g-sets of *G*. The forcing geodetic number of a graph was introduced in [3] and furthur studied in [19, 21, 22]. A *chord* of the path P is an edge joining to non-adjacent vertices of P. An $u-v$ path P is called *monophonic path* if it is a chordless path. A *monophonic set* of G is a set $S \subseteq V$ such that every vertex of *G* is contained in a monophonic path joining some pair of vertices in *M*. The *monophonic number* $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic set* or simply a m-*set* of G. The monophonic number of a graph was introduced in [6] and further studied in [2, 6, 10, 11, 12, 13, 14, 15, 19, 24]. A vertex *v* is said to be *monophonic vertex* of *G* if *v* belongs to every minimum monophonic set of *G*. A vertex *v* is an *extreme vertex* of a graph *G* if the sub graph induced by its neighbours is complete. A vertex v is said to be geodetic(monophonic) vertex if v belongs to every g-set (m-set) of G. Every extreme vertices are *geodetic(monophonic)* vertices of G. In fact there are geodetic (monophonic) vertices which are not extreme vertices of G. Let *G* be a connected graph and *S* a m-set of *G*. A subset $T \subseteq S$ is called a *forcing subset* for *S* if *S* is the unique m-set of G containing *T*. A *forcing subset* for *S* of minimum cardinality is a minimum forcing subset of *S*. The $forcing\ monophonic\ number$ of *S*, denoted by $f_m(S)$, is the cardinality of a minimum forcing subset of *S*. The *forcing monophonic number* of *G*, denoted by $f_m(G)$ is defined by $f_m(G)$ = min $\{f_m(S)\}$, where the minimum is taken over all m-sets *S* in *G*. The forcing monophonic number of a graph was introduced in [11]. The Throughout the following *G* denotes a connected graph with at least two vertices.The following theorems are used in the sequel.

Theorem 1.1. $[4, \mathbf{12}]$ If *v* is an extreme vertex of a connected graph *G*, then *v* belongs to every geodetic (monophonic) set of *G*.

Theorem 1.2. $[1, \mathbf{12}]$ For a connected graph *G*, $g(G) = p$ $(m(G) = p)$ if and only if $G = K_p$.

Theorem 1.3. $[\mathbf{3}, \mathbf{11}]$ Let G be a connected graph, then
a) $f_g(G) = 0 = f_m(G) = 0$ if and only if G has a unique minimum geodetic (monophonic) set.
b) $f_g(G) \leq g(G) - |W|$, $(f_m(G) \leq m(G) - |W|)$, where W is the set of all geodetic (monophonic) vertices of G.

Theorem 1.4. $[\mathbf{3}, \mathbf{11}]$ For the complete graph $G = K_p$, $f_g(G) = f_m(G) = 0$.

2. The Forcing Monophonic and the Forcing Geodetic Numbers of a Graph

We know that $m(G) \leq g(G)$. From the following examples, we observe that there is no relationship between $f_m(G)$ and $f_g(G)$.

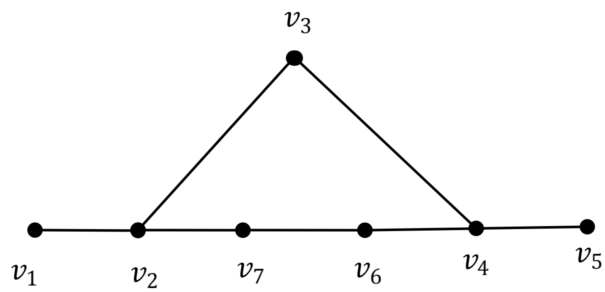


Figure 2.1

Example 2.1. For the graph *G* given in Figure 2.1, $M = \{v_1, v_5\}$ is the unique m-set of G so that $f_m(G) = 0$ and $m(G) = 2$. Also $S_1 = \{v_1, v_5, v_6\}$ and $S_2 = \{v_1, v_5, v_7\}$ are the only two g-sets of *G* such that $f_g(S_1)$ = $f_g(S_2)$ = 1 so that $f_g(G)$ = 1 and $g(G) = 3$. Thus $f_m(G) < f_g(G) < m(G) < g(G)$.

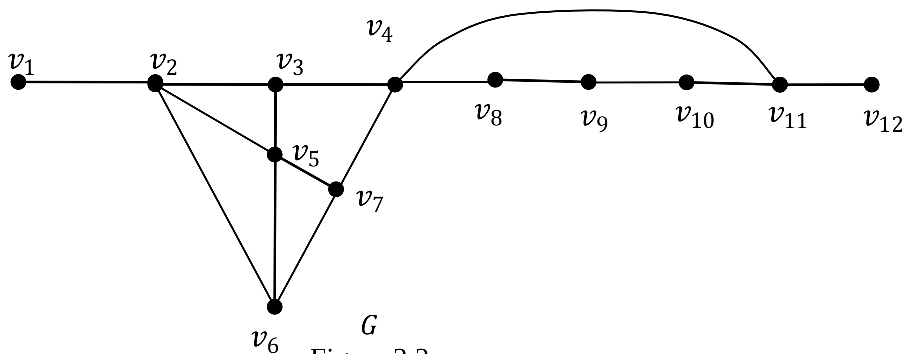


Figure 2.2

Example 2.2. For the graph *G* given in Figure 2.2, $M_1 = \{v_1, v_8, v_{12}\}$, $M_2 = \{v_1, v_9, v_{12}\}$ and $M_3 = \{v_1, v_{10}, v_{12}\}$ are the only three m-set of G so that $f_m(M_1) = f_m(M_2) = f_m(M_3) = 1$ so that $f_m(G) = 1$ and $m(G) = 3$. Also $S_1 = \{v_1, v_7, v_9, v_{12}\}$ is the unique g-set of G so that $f_g(G)$ = 0 and $g(G) = 4$. Thus $f_g(G) < f_m(G) < m(G) < g(G)$.

3. Special graphs

In this section, we present some graphs from which various graphs arising in theorem are generated using identification.

Let $P_i : u_i, v_i$ be a copy of paths on two vertices. Let G_a be the graph given in Figure 3.1 obtained from P_i $(\leq i \leq a)$ by introducing new vertices s, t and joining each u_i $(1 \leq i \leq a)$ with s and joining each v_i $(1 \leq i \leq a)$ with t and join s with t.

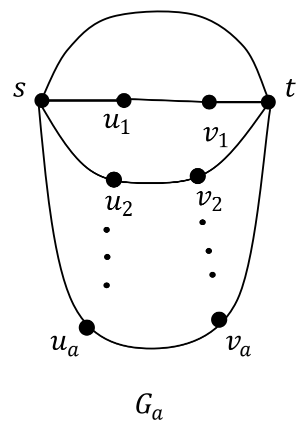


Figure 3.1

Let $P_i : n_i, p_i$ $(1 \leq i \leq b)$ be a copy of path on two vertices and $P : l, m, n$ be a path on three vertices. Let Z_b be the graph given in Figure 3.2 obtained from P_i $(1 \leq i \leq b)$ and P by joining each n_i $(1 \leq i \leq b)$ with l, each p_i $(1 \leq i \leq b)$ with q.

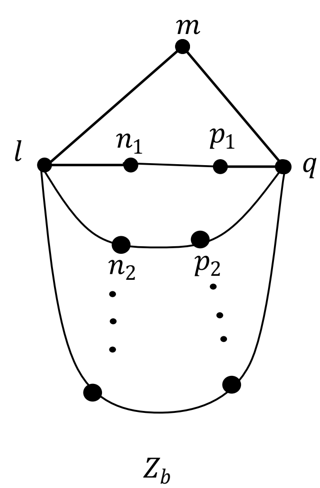
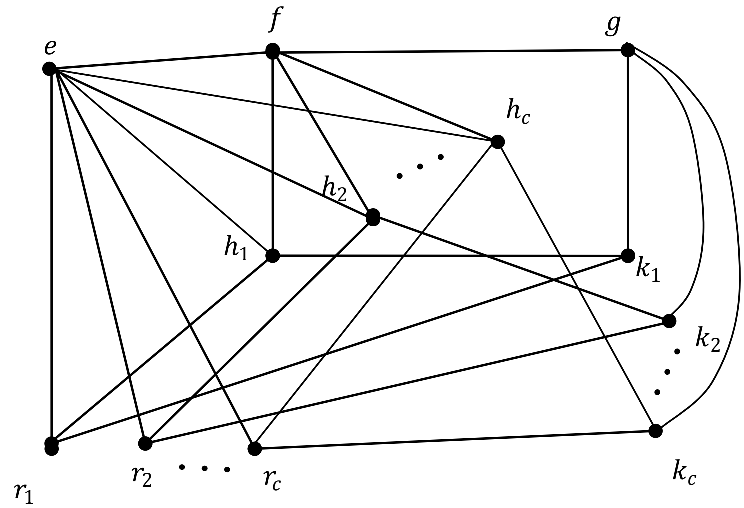


Figure 3.2

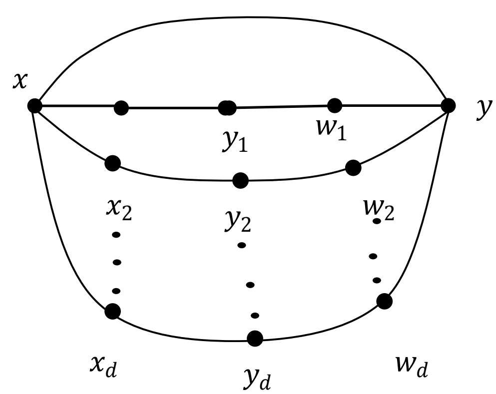
Let $P_i : r_i, h_i, k_i$ $(1 \leq i \leq c)$ be a copy of path on three vertices and let P: e, f, g be a path on three vertices. Let H_c be the graph given in Figure 3.3 obtained from P_i $(1 \leq i \leq c)$ and P by joining e and f with each h_i and r_i $(1 \leq i \leq c)$, joining g with each k_i $(1 \leq i \leq c)$, joining h_i $(1 \leq i \leq c)$ with k_i $(1 \leq i \leq c)$, and joining r_i $(1 \leq i \leq c)$ with k_i $(1 \leq i \leq c)$.



H_c

Figure 3.3

Let U_i: x_i, y_i, w_i $(1 \leq i \leq d)$ be the path on three vertices. Let R_a be the graph given in Figure 3.4 obtained from U_i $(1 \leq i \leq d)$ by adding new vertices u and v by joining u with v and joining each x_i $(1 \leq i \leq d)$ with u and joining each w_i $(1 \leq i \leq d)$ with v.



R_d

Figure 3.4

4. Some realization results

Theorem 4.1. For every pair a, b of integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph *G* such that $f_m(G) = f_g(G) = 0$, $m(G) = a$ and $g(G) = b$.

Proof. If $a = b$, let $G = K_a$. Then by Theorem 1.2, $m(G) = g(G) = a$. Also by Theorem 1.3(a),$f_m(G) = f_g(G) = 0$. For $1 \leq a < b$, let G be the graph obtained from H_{b-a} by adding new vertices $x, z_1, z_2, ..., z_{a-1}$ and joining the edges $xe, gz_1, gz_2, ..., gz_{a-1}$. Let $Z = \{x, z_1, z_2, ..., z_{a-1}\}$ be the set of all end-vertices of G. Then it is clear that Z is a monophonic set of G and so by Theorem 1.1, Z is the unique m-set of G so that $m(G) = a$ and hence by Theorem 1.3(a), $f_m(G) = 0$. Since the vertices h_i, k_i and r_i $(1 \leq i \leq b-a)$ does not lie on any geodesic joining a pair of vertices in Z, we see that Z is not a geodetic set of G. It is easily verified that every g-set of G contains each h_i $(1 \leq i \leq b-a)$ and so $g(G) \geq b$. Now it is easily seen that $W = Z \cup \{h_1, h_2, ..., h_{b-a}\}$ is the unique g-set of G and hence by Theorem 1.1 and Theorem 1.3(a) $g(G) = b$ and $f_g(G) = 0$. □

Theorem 4.2. For every integers a, b and c with $0 \leq a < b < c$ and $c > a + b$, there exists a connected graph G such that $f_m(G) = 0$, $f_g(G) = a$, $m(G) = b$ and $g(G) = c$.

Proof. **Case 1.** $a = 0$. Then the graph G constructed in Theorem 4.1 satisfies the requirements of this theorem.

Case 2. $a \geq 1$. Let G be the graph obtained from Z_a and $H_{c-(a+b)}$ by identifying the vertex q of Z_a and e of $H_{c-(a+b)}$ and then adding new vertices $x, z_1, z_2, ..., z_{b-1}$ and joining the edges $xl, gz_1, gz_2, ..., qz_{b-1}$. It is clear that Z is a monophonic set of G and by Theorem 1.1, Z is the unique m-set of G so that $m(G) = b$ and hence by Theorem 1.3(a), $f_m(G) = 0$. Next we show that $g(G) = c$. Let S be any geodetic set of G. Then by Theorem 1.1, $Z \subseteq S$. It is clear that Z is not a geodetic set of G. For $1 \leq i \leq a$, let $Q_i = \{n_i, p_i\}$. We observed that every g-set of G must contain at least one vertex from each Q_i $(1 \leq i \leq a)$ and each h_i $(1 \leq i \leq c-b-a)$ so that $g(G) \geq b + a + c - a - b = c$. Now $W = Z \cup \{h_1, h_2, ..., h_{c-a-b}\} \cup \{n_1, n_2, ..., n_a\}$ is a geodetic set of G so that $g(G) \geq b + a + c - a - b = c$. Thus $g(G) = c$. Since every g- set contains $W_1 = Z \cup \{h_1, h_2, ..., h_{c-a-b}\}$ it follows from that from Theorem 1.3 (b) that $f_g(G) \leq g(G) - \mid W_1 \mid = c - (c-a) = a$. Now, since $g(G) = c$ and every g-set of G contains W_1, it is easily seen that every g-set S is of the form $W_1 \cup \{d_1, d_2, ..., d_a\}$ where $d_i \in Q_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $\mid T \mid < a$. Then it is clear that there exists some j such that $T \cap Q_j = \Phi$, which shows that $f_g(G) = a$. □

Theorem 4.3. For every integers a, b and c with $0 \leq a < b \leq c$ and $b > a + 1$ there exists a connected graph G such that $f_g(G) = 0$, $f_m(G) = a$, $m(G) = b$ and $g(G) = c$.

Proof. **Case 1**. $a = 0$. Then the graph G constructed in Theorem 4.1 satisfies the requirements of this theorem.

Case 2. $a \geq 1$.

Subcase 2a. $b = c$. Let G be the graph obtained from R_a by adding new vertices $x, z_1, z_2, \ldots,$ z_{b-a-1} and joining the edges $xu, vz_1, vz_2, \ldots, vz_{b-a-1}$. Let $Z = \{x, z_1, z_2, \ldots, z_{b-a-1}\}$ be the set of all end-vertices of G. Let S be any geodetic set of G. Then by Theorem 1.1, $Z \subseteq S$. It is clear that Z is not a geodetic set of G. For $1 \leq i \leq a$, let $H_i = \{x_i, y_i, w_i\}$. We observe that every g-set of G must contain only the vertex y_i from each H_i $(1 \leq i \leq a)$ and so $g(G) \geq b - a + a = b$. Now $S = Z \cup \{y_1, y_2, y_3, \ldots, y_a\}$ is a geodetic set of G so that $g(G) \leq b - a + a = b$. Thus $g(G) = b$. Also it is easily seen that W is the unique g-set of G and so $f_g(G) = 0$. Now it is clear that Z is not a monophonic set of G. We observe that every m-set of G must contain at least one vertex from each H_i $(1 \leq i \leq a)$. Hence by Theorem 1.1, $m(G) \geq b - a + a = b$. Now

$W_1 = Z \cup \{y_1, y_2, y_3, \ldots, y_a\}$ is a monophonic set of G so that $m(G) \leq b - a + a = b$. Thus $m(G) = b$. Next we show that $f_m(G) = a$. Since every m-set contains Z, it follows from Theorem 1.3 (b) that $f_m(G) \leq m(G) - \mid Z \mid = b - (b - a) = a$. Now, since $m(G) = b$ and every m-set of G contains Z, it is easily seen that every m-set S is of the form $Z \cup \{d_1, d_2, d_3, \ldots, d_a\}$, where $d_i \in H_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $\mid T \mid < a$. Then it is clear that there exists some j such that $T \cap H_j = \Phi$, which shows that $f_m(G) = a$.
Subcase 2b. $b < c$. Let G be the graph obtained from R_a and H_{c-b} by identifying the vertex v of R_a and g of H_{c-b} and then adding the new vertices $x, z_1, z_2, \ldots, z_{b-a-1}$ and joining the edges $xu, gz_1, gz_2, \ldots, gz_{b-a-1}$. Let $Z = \{x, z_1, z_2, \ldots, z_{b-a-1}\}$ be the set of end vertices of G. Let S be any geodetic set of G. Then by Theorem 1.1 $Z \in S$. It is clear that Z is not a geodetic set of G. For $1 \leq i \leq a$, let $H_i = \{x_i, y_i, w_i\}$. We observe that every g-set of G must contain only the vertex y_i $(1 \leq i \leq a)$ from each H_i $(1 \leq i \leq a)$ and each h_i $(1 \leq i \leq c - b)$ and so $g(G) \geq b - a + a + c - b = c$ Now $W = Z \cup \{y_1, y_2, y_3, \ldots, y_a\} \cup \{h_1, h_2, h_3, \ldots, h_{c-b}\}$ is a geodetic set of G so that $g(G) \leq b - a + a + c - b = c$. Thus $g(G) = c$. Also it is easily seen that W is the unique g-set of G and so $f_g(G) = 0$. It is clear that Z is not a monophonic set of G. We observe that every m-set of G must contain at least one vertex from each H_i $(1 \leq i \leq a)$ and so $m(G) \geq b - a + a = b$. Now, $S_1 = Z \cup \{y_1, y_2, y_3, \ldots, y_a\}$ is a monophonic set of G so that $m(G) \leq b - a + a = b$. Thus $m(G) = b$. Next we show that $f_m(G) = a$. Since every m-set contains Z, it follows from Theorem 1.3 (b) that $f_m(G) \leq m(G) - \mid Z \mid = b - (b - a) = a$. Now, since $m(G) = b$ and every m-set of G contains Z, it is easily seen that every m-set S is of the form $Z \cup \{d_1, d_2, d_3, \ldots, d_a\}$, where $d_i \in H_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $\mid T \mid < a$. Then it is clear that there exists some j such that $T \cap H_j = \Phi$, which shows that $f_m(G) = a$. □

Theorem 4.4. For every pair a, b and c of integers with $0 \leq a \leq b \leq c$, $b > a + 1$ there exists a connected graph G such that $f_g(G) = f_m(G) = a$, $m(G) = b$ and $g(G) = c$.

Proof. **Case 1.** $a = 0$, then the graph G constructed in Theorem 4.1 satisfies the requirements of this theorem.
Case 2. $a \geq 1$,
Subcase 2a. $b = c$. Let G be the graph obtained from G_a by adding new vertices $x, z_1, z_2, \ldots,$ z_{b-a-1} and joining the edges $xs, tz_1, tz_2, \ldots, tz_{b-a-1}$. Let $Z = \{x, z_1, z_2, \ldots, z_{a-b-1}\}$ be the set of end-vertices of G. First we show that $m(G) = b$. Let M be any monophonic set of G. Then by Theorem 1.1, $Z \subseteq M$. It is clear that Z is not a monophonic set of G. Let $F_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$. We observe that every m-set of G must contain at least one vertex from each F_i $(1 \leq i \leq a)$. Thus $m(G) \geq b - a + a = b$. On the other hand since the set $W = Z \cup \{v_1, v_2, \ldots, v_a\}$ is a monophonic set of G, it follows that $m(G) \leq \mid W \mid = b$. Hence $m(G) = b$. Next we show that $f_m(G) = a$. By Theorem 1.1, every monophonic set of G contains Z and so it follows from Theorem 1.3(b) that $f_m(G) \leq m(G) - \mid Z \mid = a$. Now, since $m(G) = b$ and every m-set of G contains Z, it is easily seen that every m-set S is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in F_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $\mid T \mid < a$. Then it is clear that there exists some j such that $T \cap H_j = \Phi$, which shows that $f_m(G) = a$. By similar way we can prove $g(G) = b$ and $f_g(G) = a$.
Subcase 2b. $b < c$. Let G be the graph obtained from G_a and H_{c-b} by identifying the vertex t of

G_a and the vertex e of H_{c-b} and then adding the new vertices $x, z_1, z_2, \ldots, z_{b-a-1}$ and joining the edges $xs, gz_1, gz_2, \ldots, gz_{b-a-1}$.
First we show that $m(G) = b$. Let $Z = \{z_1, z_2, \ldots, z_{b-a-1}\}$ be the set of all end-vertices of G. Since the vertices u_i, v_i do not lie on any monophonic path joining a pair of vertices of Z, it is clear that Z is not a monophonic set of G. Let $F_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$. We observe that every m-set of G must contain at least one vertex from each F_i $(1 \leq i \leq a)$. Thus $m(G) \geq b - a + a = b$. On the other hand since the set $W = Z \cup \{v_1, v_2, v_3, \ldots, v_a\}$ is a monophonic set of G, it follows that $m(G) \leq |W| = b$. Hence $m(G) = b$. Next, we show that $f_m(G) = a$. By Theorem 1.1, every monophonic set of G contains Z and so it follows from Theorem 1.3 (b) that $f_m(G) \leq m(G)- \mid Z \mid= a$. Now, since $m(G) = b$ and every m-set of G contains Z, it is easily seen that every m-set S is of the form $Z \cup \{c_1, c_2, c_3, \ldots, c_a\}$, where $c_i \in F_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $\mid T \mid< a$. Then it is clear that there exists some j such that $T \cap H_j = \Phi$, which shows that $f_m(G) = a$. Next we show that $g(G) = c$. Since the vertices u_i, v_i, h_i $(1 \leq i \leq a)$ do not lie on any geodesic joining a pair of vertices of Z, it is clear that Z is not a geodetic set of G. We observe that every g-set of G must contain each H_i $(1 \leq i \leq a)$ and each h_i $(1 \leq i \leq c-b)$ so that $g(G) \geq b - a + a + c - b = c$. On the other hand, since the set $S_1 = Z \cup \{h_1, h_2, h_3, \ldots, h_{c-b}\} \cup \{u_1, u_2, \ldots, u_a\}$ is a geodetic set of G, so that $g(G) \leq\mid S_1 \mid= c$. Hence $g(G) = c$. Next we show that $f_g(G) = a$. By Theorem 1.1, every geodetic set of G contains $W_1 = Z \cup \{h_1, h_2, h_3, \ldots, h_{c-b}\}$ and so it follows from Theorem 1.3(b) that $f_g(G) \leq g(G)- \mid W_1 \mid= a$. Now, since $g(G) = c$ and every g-set of G contains Z, it is easily seen that every g-set S is of the form $W_1 \cup \{c_1, c_2, c_3, \ldots, c_a\}$, where $c_i \in F_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $\mid T \mid< a$. Then it is clear that there exists some j such that $T \cap H_j = \Phi$, which shows that $f_g(G) = a$. This is true for all g-sets of G so that $f_g(G) = a$. □

Theorem 4.5. For every integers *a,b,c* and *d* with $2 \leq c < d$, $0 \leq a \leq b \leq d$ and $d > c - a + b$, there exists a connected graph *G* such that $f_m(G) = a$, $f_g(G) = b$, $m(G) = c$ and $g(G) = d$.

Proof. **Case 1.** $a = b = 0$. Then the graph G constructed in Theorem 4.1 satisfies the requirements of this theorem.
Case 2. $a = 0$, $b > 1$. Then the graph G constructed in Theorem 4.2 satisfies the requirements of this theorem.
Case 3. $1 \leq a = b$. Then the graph G constructed in Theorem 4.4 satisfies the requirements of this theorem.
Case 4. $1 \leq a < b$. Let G_1 be the graph obtained from G_a and Z_{b-a} by identifying the vertex t of G_a and the vertex l of Z_{b-a}. Now let G be the graph obtained from G_1 and $H_{d-(c-a+b)}$ by identifying the vertex q of G_1 and the vertex e of $H_{d-(c-a+b)}$ and adding new vertices $x, z_1, z_2, ..., z_{c-a-1}$ and joining the edges $xs, gz_1, gz_2, ..., gz_{c-a-1}$. Let $Z = \{x, z_1, z_2, \ldots, z_{c-a-1}\}$ be the set of end vertices of *G*. For $1 \leq i \leq a$ let $F_i = \{u_i, v_i\}$. It is clear that any m-set is of the form $S = Z \cup \{c_1, c_2, c_3, \ldots, c_a\}$ where $c_i \in F_i$ $(1 \leq i \leq a)$. Then as in earlier theorems it can be seen that $f_m(G) = a$ and $m(G) = c$. For $1 \leq i \leq a$ let $Q_i = \{n_i, p_i\}$. It is clear that any *g*-set is of the form $W = Z \cup \{h_1, h_2, h_3, \ldots, h_{d-(c-a+b)}\} \cup \{c_1, c_2, c_3, \ldots, c_a\} \cup \{d_1, d_2, d_3, \ldots, d_{b-a}\}$, where $c_i \in F_i$ $(1 \leq i \leq a)$ and $d_j \in Q_j$ $(1 \leq j \leq b-a)$. Then as in earlier theorems it can be seen that $f_g(G) = b$ and $g(G) = d$. □

Theorem 4.6. For every integers a, b, c and d with $0 \leq a \leq b < c \leq d$ and $c \geq b+1$ and $c, d \geq 2$ there exists a connected graph G such that $f_g(G) = a$, $f_m(G) = b$, $m(G) = c$ and $g(G) = d$.

Proof. **Case 1.** $a = b = 0$. Then the graph G constructed in Theorem 4.1 satisfies the requirements of this theorem.
Case 2. $a = 0, b \geq 1$. Then the graph *G* constructed in Theorem 4.2 satisfies the requirements of this theorem.
Case 3. $1 \leq a = b$. Then the graph G constructed in Theorem 4.4 satisfies the requirements of this theorem.
Case 4. $1 \leq a < b$.
Subcase 4a. $c = d$. Let G be the graph obtained from G_a and R_{b-a} by identifying the vertex t of G_a and the vertex q of R_{b-a} and then adding the new vertices $x, z_1, z_2, \ldots, z_{c-b-1}$ and joining the edges $xs, qz_1, qz_2, \ldots, qz_{c-b-1}$. First we show that $m(G) = c$. Let $Z = \{x, z_1, z_2, \ldots, z_{c-b-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$ and $H_i = \{x_i, y_i, w_i\}$ $(1 \leq i \leq b-a)$. It is clear that any m-set of G is of the form $S = Z \cup \{c_1, c_2, c_3, \ldots, c_a\} \cup \{d_1, d_2, d_3, \ldots, d_{b-a}\}$ where $c_i \in F_i$ $(1 \leq i \leq a)$ and $d_j \in H_j$ $(1 \leq j \leq b-a)$. Then as in earlier theorems it can be seen that $f_m(G) = b$ and $m(G) = c$. It is clear that any g-set is of the form $W = Z \cup \{y_1, y_2, y_3, \ldots, y_{b-a}\} \cup \{c_1, c_2, c_3, \ldots, c_a\}$, where $c_i \in F_i$ $(1 \leq i \leq a)$. Then as in earlier theorems it can be seen that $f_g(G) = a$ and $m(G) = c$.
Subcase 4b. $c < d$. Let G_1 be the graph obtained from G_a and R_{b-a} by identifying the vertex t of G_a and the vertex v of R_{b-a}. Now let G be the graph obtained from G_1 and Z_{d-c} by identifying the vertex q of G_1 and the vertex l of Z_{d-c} and then adding new vertices $x, z_1, z_2, \ldots, z_{c-b-1}$ and joining the edges $xs, qz_1, qz_2, \ldots, qz_{c-b-1}$. Let $Z = \{x, z_1, z_2, \ldots, z_{c-b-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$ and $H_i = \{x_i, y_i, w_i\}$ $(1 \leq i \leq b-a)$. It is clear that any m-set of G is of the form $S = Z \cup \{c_1, c_2, c_3, \ldots, c_a\} \cup \{d_1, d_2, d_3, \ldots, d_{b-a}\}$ where $c_i \in F_i$ $(1 \leq i \leq a)$ and $d_j \in H_j$ $(1 \leq j \leq b-a)$. Then as in earlier theorems it can be seen that $f_m(G) = b$ and $m(G) = c$. It is clear that any g-set is of the form $W = Z \cup \{y_1, y_2, y_3, \ldots, y_{b-a}\} \cup \{h_1, h_2, h_3, \ldots, h_{d-c}\} \cup \{c_1, c_2, c_3, \ldots, c_a\}$ where $c_i \in F_i$ $(1 \leq i \leq a)$. Then as in earlier theorems it can be seen that $f_g(G) = a$ and $g(G) = d$. □

In the realiztion results we have given some restrictions on the parameters. So we leave the following as open question.

Problem 1. *For any four positive integers a,b,c and d with $a \geq 0$, $b \geq 0$ and $2 \leq c \leq d$, does there exists a connected graph G with $f_m(G) = a$, $f_g(G) = b$, $m(G) = c$ and $g(G) = d$.*

5. The Upper Forcing Monophonic number of a graph

In [25], P. Zhang introduced the concept of the upper geodetic number of a graph. In the similar manner we define the upper forcing monophonic number of a graph as follows.

Definition 5.1. Let G be a connected graph and S a m-set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique m-set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of *S*. The $forcing\ monophonic\ number$ of *S*, denoted by $f_m(S)$, is the cardinality of a minimum forcing subset of *S*. The *forcing monophonic number* of *G*, denoted by $f_m(G)$ is defined by $f_m(G)$ = min $\{f_m(S)\}$, where the minimum is taken over all m-set S in G and the upper forcing monophonic number of G, denoted by $f_m^+(G) = max\{f_m(S)\}$, where the maximum is taken over all m-sets *S* in *G*.

Theorem 5.2. For every connected graph *G*, $0 \leq f_m(G) \leq f_m^+(G) \leq m(G)$.

Example 5.3. The bounds in Theorem 5.2 is sharp. For $G = K_{1,p-1}$, $f_m(G) = 0$. For $G = C_5$, $f_m(G) = f_m^+(G) = 2$. Also the inequalities in Theorem 5.2 can be strict. For the graph G given in Figure 5.1, $M_1 = \{v_1, v_4, v_5\}$, $M_2 = \{v_1, v_4, v_6\}$ and $M_1 = \{v_1, v_3, v_5\}$ are only three m-sets of G so that $f_m(M_1) = 2$, $f_m(M_2) = 1$ and $f_m(M_3) = 2$ so that $f_m(G) = 2$, $f_m^+(G) = 2$ and $m(G) = 3$. Thus $0 < f_m(G) < f_m^+(G) < m(G)$.

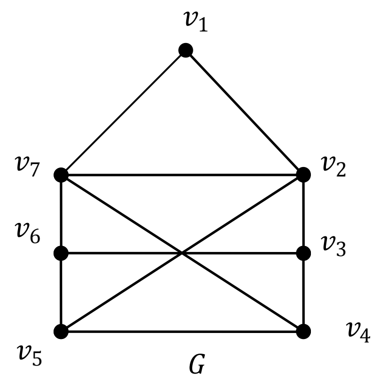


Figure 5.1

So we leave the following as a open question.

Problem 2. *For any three positive integers a,b and c with* $0 \leq a \leq b \leq c$*, does there exists a connected graph G with* $f_m(G) = a$*,* $f_m^+(G) = b$ *and* $m(G) = c$*.*

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