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On super (a, d)-edge antimagic total labeling of branched-prism graph

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Abstract

Let H be a branched-prism graph, denoted by $H = (C_m \times P_2) \odot \overline{K}_n$ for odd m, $m \geq 3$ and $n \geq 1$. This paper considers about the existence of the super (a, d)-edge antimagic total labeling of H for some positive integer a and some non-negative integer d.

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1. Introduction

In [5], Hartsfield and Ringel gave the concept of antimagic labeling of a graph. Let G be an arbitrary graph G on p vertices and q edges. Graph G is called antimagic if its edges are labeled with $1, 2, \cdots, q$ such that all the vertex weights are pairwise distinct. Next, for some integers $a > 0$ and $d \geq 0$, Bodendick and Walther [3] introduced the concept of (a, d)-antimagic labeling as an edge labeling such that the vertex weights form an arithmetic progression starting from a and having a common difference d. Moreover, Simanjuntak *et al.* [6] defined an (a, d)-edge antimagic vertex labeling of G as a mapping $f : V(G) \to \{1, 2, \cdots, p\}$ such that the set of edge weights $W_1 = \{f(u) + f(v) \mid uv \in E(G)\}$ can be written as $\{a, a + d, a + 2d, \cdots, a + (q-1)d\}$, for some non-negative integers a and d. A mapping $g : V(G) \cup E(G) \to \{1, 2, \cdots, p + q\}$ such that

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the set of edge weights $W_2 = \{g(u) + g(v) + g(uv) \mid uv \in E(G)\}$ form an arithmetic progression $\{a, a + d, a + 2d, \cdots, a + (q - 1)d\}$, for $a > 0$ and $d \geq 0$, is called an (a, d)-edge antimagic total labeling of G. If $d = 0$ then the $(a, 0)$-antimagic labeling becomes the magic labeling with magic constant a. Some previous results on magic and antimagic labeling are listed in a book by Baca and Miller [2] and also in an updated survey by Gallian [4].

Sugeng *et a*l. [7] determined the super (a, d)-edge antimagic total labeling (super (a, d)-EAMTL) of a generalized prism graph $C_m \times P_r$ for $m \geq 3$ and $r \geq 2$. In [1], Azizu et *al.* defined the branched-prism graph, denoted by $H = (C_m \times P_2) \odot \overline{K}_n$, for $m \geq 3$ and $n \geq 1$, where $\overline{K}_n$ denotes the complement of a complete graph on n vertices. They also determined the existence of a super edge magic labeling (super EMTL) of the branched-prism graph. In this paper, it will be shown that H admits a super (a, d)-edge antimagic total labeling for odd m, $m \geq 3$ and $n \geq 1$.

2. The Branched-Prism Graph and Its Super (a, d)-Edge Antimagic Labeling

Azizu *et al*. [1] gave the definition of branched-prism graph as follows. The graph is constructed from the corona operation between prism graph $C_m \times P_2$ and the complement of a complete graph $\overline{K}_n$ on n vertices, denoted by $H = (C_m \times P_2) \odot \overline{K}_n$, for $m \geq 3$ and $n \geq 1$. The vertex set and edge set of H are defined as follows.

$$\begin{aligned} V(H) &= \{v_{i,j}, v_{i,j,k} \mid 1 \leq i \leq 2, 1 \leq j \leq m, 1 \leq j \leq n\}, \\ E(H) &= \{v_{i,j}v_{i,j+1} \mid 1 \leq i \leq 2, 1 \leq j \leq m-1\} \cup \{v_{i,m}v_{i,1} \mid 1 \leq i \leq 2\}, \\ &\quad \cup \{v_{1,j}v_{2,j} \mid 1 \leq j \leq m-1\} \cup \{v_{1,m}v_{2,m}\} \\ &\quad \cup \{v_{i,j}v_{i,j,k}, v_{i,m}v_{i,m,k} \mid 1 \leq i \leq 2, 1 \leq j \leq m-1, 1 \leq k \leq n\}. \end{aligned}$$

It is clear that H has $p = 2mn + 2m$ vertices and $q = 2mn + 3m$ edges. Graph H is given in Figure 1.

The following theorem gives the upperbound of the difference d in the super (a, d)-EAMTL of H.

Theorem 2.1. *Let $H = (C_m \times P_2) \odot \overline{K_n}$ be the branched-prism graph on $2mn + 2m$ vertices and $2mn + 3m$ edges. If H admits the super (a, d)-edge antimagic total labeling then $d \leq 2$.*

Proof. Suppose that H has a super (a, d)-EAMTL, defined by $f : V(H) \cup E(H) \rightarrow \{1, 2, \cdots, 4mn + 5m\}$. The set of edge weight can be written as $\{a, a + d, \cdots, a + (q - 1)d\}$, where $q = 2mn + 3m$. The minimum possible edge-weight of this labeling is $1 + 2 + (2mn + 2m + 1) = 2mn + 2m + 4$, and the maximum possible edge-weight is $(2mn + 2m - 1) + (2mn + 2m) + ((2mn + 2m) + (2mn + 3m)) = 3(2mn + 2m) + (2mn + 3m) - 1 = 8mn + 9m - 1$. Therefore,

$$a \geq 2mn + 2m + 4, \quad (1)$$

$$a + (q - 1)d = a + (2mn + 3m - 1)d \leq 8mn + 9m - 1. \quad (2)$$

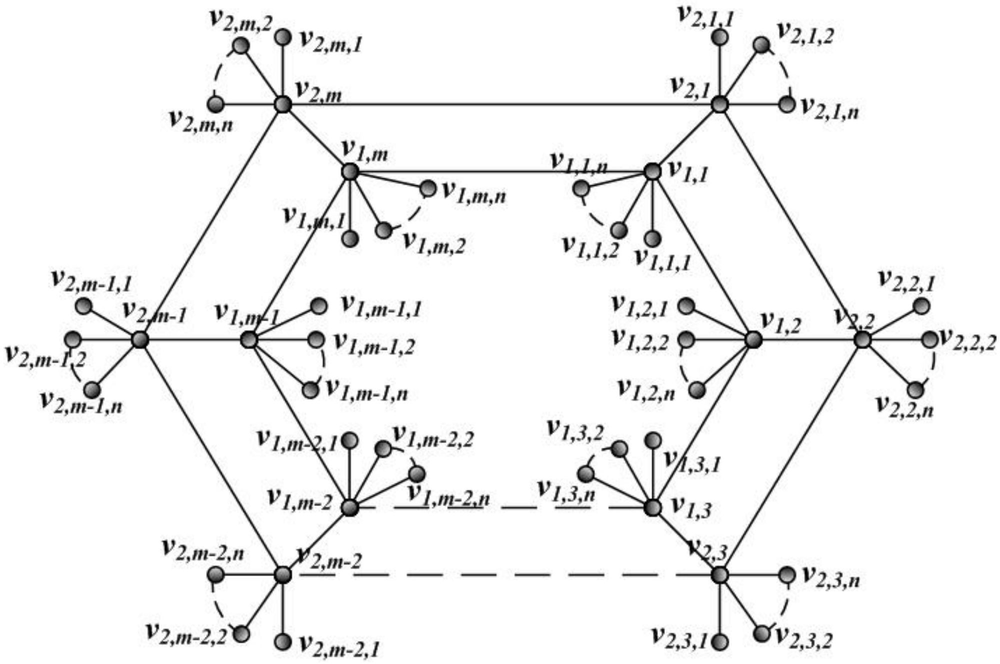

Figure 1. [1] The branched-prism graph $H = (C_m \times P_2) \odot \overline{K_n}$

From (1) and (2), we have the following inequality:

$$\begin{aligned} 2mn + 2m + 4 + (2mn + 3m - 1)d &\leq 8mn + 9m - 1, \\ d &\leq \frac{8mn + 9m - 1 - (2mn + 2m + 4)}{2mn + 3m - 1}, \\ &= \frac{6mn + 7m - 5}{2mn + 3m - 1}. \end{aligned}$$

For $m \geq 3$ and $n \geq 1$, it is clear that $d < 3$. Therefore, if H admits the super (a, d)-EAMTL then $d \in \{0, 1, 2\}$. □

The following theorem gives the super (a, d)-EAMTL of H for $d = 1$ and $d = 2$. The super $(a, 0)$-EAMTL of H, or the super EMTL of H, has been obtained in [1].

Theorem 2.2. *Let $H = (C_m \times P_2) \odot \overline{K_n}$ be the branched-prism graph on $2mn + 2m$ vertices and $2mn + 3m$ edges. For odd m, $m \geq 3$ and $n \geq 1$, there exist a super $(a_1, 1)$-edge antimagic total labeling and a super $(a_2, 2)$-edge antimagic total labeling of H, where $a_1 = 4mn + 4m + 2$ and $a_2 = 3mn + 2m + \frac{(m+1)}{2} + 2$.*

Proof. Let $H = (C_m \times P_2) \odot \overline{K_n}$ be the branched-prism graph on $2mn + 2m$ vertices and $2mn + 3m$ edges, for odd m, $m \geq 3$ and $n \geq 1$. First, define the vertex labeling $f : V(H) \rightarrow$

$\{1, 2, \cdots, 2mn + 2m\}$ as follows.

$$f(v_{1,i,j}) = \begin{cases} \dfrac{i+1}{2} + m(j-1), & \text{for odd } i,\ 1 \le i \le m, 1 \le j \le n, \\ \dfrac{m+i+1}{2} + m(j-1), & \text{for even } i,\ 2 \le i \le m-1, 1 \le j \le n. \end{cases}$$

$$f(v_{1,i}) = \begin{cases} mn + \dfrac{i}{2}, & \text{for even } i,\ 2 \le i \le m-1, \\ mn + \dfrac{m+i}{2}, & \text{for odd } i,\ 1 \le i \le m. \end{cases}$$

$$f(v_{2,i}) = \begin{cases} mn + 2m, & \text{for } i = 1, \\ mn + m + \dfrac{i-1}{2}, & \text{for odd } i,\ 3 \le i \le m, \\ mn + m + \dfrac{m+i-1}{2}, & \text{for even } i,\ 2 \le i \le m-1. \end{cases}$$

$$f(v_{2,i,j}) = \begin{cases} mn + 2m + \dfrac{m-1}{2} + m(j-1), & \text{for } i = 1,\ 1 \le j \le n, \\ mn + 3m + m(j-1), & \text{for } i = 2,\ 1 \le j \le n, \\ mn + 2m + \dfrac{i-2}{2} + m(j-1), & \text{for even } i,\ 4 \le i \le m-1,\ 1 \le j \le n, \\ mn + 2m + \dfrac{m+i-2}{2} + m(j-1), & \text{for odd } i,\ 3 \le i \le m,\ 1 \le j \le n. \end{cases}$$

Denote $S = \{f(x) + f(y) \mid xy \in E(H)\}$ as the set of edge weights of the vertex labeling of H. Therefore, $S = \{mn+1+\frac{(m+1)}{2}, mn+\frac{(m+1)}{2}+2, \cdots, 3mn+3m+\frac{(m+1)}{2}-1, 3mn+3m+\frac{(m+1)}{2}\}$.

Next, define the edge labeling $f : E(H) \to \{2mn+2m+1, 2mn+2m+2, \cdots, 4mn+5m\}$ as follows. The set of edge weights of the total labeling is denoted by $W = \{f(x)+f(y)+f(xy) \mid xy \in E(H)\}$. It can be seen that $W = \{s + f(xy) \mid xy \in E(H), s \in S\}$. Consider the following cases.

Case 1. $d = 2$.

Define the minimum edge weight as

$$\begin{aligned} a &= \min\{f(xy) \mid xy \in E(H)\} + \min\{s \mid s \in S\} \\ &= (2mn + 2m + 1) + (mn + 1 + \frac{(m+1)}{2}) \\ &= 3mn + 2m + \frac{(m+1)}{2} + 2. \end{aligned}$$

By choosing this minimum value of edge weight, we have the difference $d = 2$.

Case 2. $d = 1$.

Let $s \in S$. Define the edge labeling of H as follows.

$$f(xy) = \begin{cases} s, & \text{for } 2mn + 2m + 1 \le s \le 3mn + 3m + \dfrac{m+1}{2}, \\ 2mn + 3m + s, & \text{for } mn + 1 + \dfrac{m+1}{2} \le s \le 2mn + 2m. \end{cases}$$

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By defining this edge labeling, the minimum edge-weight is

$$a = s + f(xy) = (2mn + 2m + 1) + (2mn + 2m + 1) = 4mn + 4m + 2.$$

Therefore, there exist a super $(a_1, 1)$-EAMTL and a super $(a_2, 2)$-EAMTL of H, where $a_1 = 4mn + 4m + 2$ and $a_2 = 3mn + 2m + \frac{(m+1)}{2} + 2$. □

In Figure 2 we give a super $(75, 2)$-EAMTL of $(C_5 \times P_2) \odot \overline{K_4}$. The red labels are for the vertices, while the blue ones are for the edges.

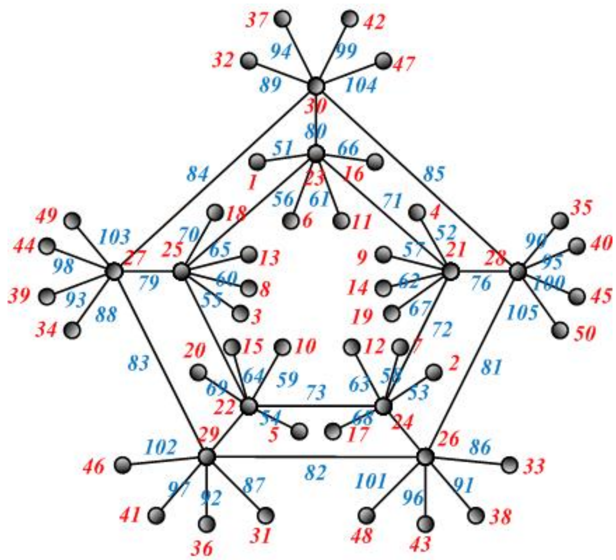

Figure 2. A Super $(75, 2)$-EAMTL of $(C_5 \times P_2) \odot \overline{K_4}$

3. Conclusion

This paper shows that the branched-prism graph $H = (C_m \times P_2) \odot \overline{K_n}$ admits a super $(4mn + 4m+2, 1)$-EAMTL and $(3mn+2m+\frac{(m+1)}{2}+2, 2)$-EAMTL for odd $m,\ m \geq 3, n \geq 1$. Combining this result with [1], we have the super (a, d)-EAMTL of the branched-prism graph for $d \in \{0, 1, 2\}$.

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