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On generalized composed properties of generalized product graphs

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Abstract

A property $\mathcal{P}$ is defined to be a nonempty isomorphism-closed subclass of the class of all finite simple graphs. A nonempty set S of vertices of a graph G is said to be a $\mathcal{P}$-set of G if $G[S] \in \mathcal{P}$. The maximum and minimum cardinalities of a $\mathcal{P}$-set of G are denoted by $M_{\mathcal{P}}(G)$ and $m_{\mathcal{P}}(G)$, respectively. If S is a $\mathcal{P}$-set such that its cardinality equals $M_{\mathcal{P}}(G)$ or $m_{\mathcal{P}}(G)$, we say that S is an $M_{\mathcal{P}}$-set or an $m_{\mathcal{P}}$-set of G, respectively. In this paper, we not only define six types of property $\mathcal{P}$ by the using concepts of graph product and generalized graph product, but we also obtain $M_{\mathcal{P}}$ and $m_{\mathcal{P}}$ of product graphs in each type and characterize its $M_{\mathcal{P}}$-set.

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1. Introduction

Throughout this paper, all graphs are considered to be finite and simple. Let $G = (V, E)$ be a graph. For a subset S of V, the induced subgraph of S will be denoted by $G[S]$. A subgraph H of G is said to be spanning whenever $V(H) = V(G)$. We remark that a graph without edges is called an empty graph. For other graph terminologies and notations, we refer the reader to [5].

Given two graphs G and H; a product of G and H, denoted by $G * H$, is a graph with the vertex set $V(G) \times V(H)$. Many definitions exist that are known as the product of G and H, especially the Cartesian, the direct, the strong and the lexicographic products. The graph $G * H$ is called a *Cartesian product* of G and H if two vertices (v_1, h_1) and (v_2, h_2) are adjacent whenever $v_1v_2 \in E(G)$ and $h_1 = h_2$, or $v_1 = v_2$ and $h_1h_2 \in E(H)$. The graph $G * H$ is called a *direct*

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product of G and H if two vertices (v_1, h_1) and (v_2, h_2) are adjacent whenever $v_1v_2 \in E(G)$ and $h_1h_2 \in E(H)$. The graph $G * H$ is called a *strong product* of G and H if it is a Cartesian or direct product. The graph $G * H$ is called a *lexicographic product* of G and H if two vertices (v_1, h_1) and (v_2, h_2) are adjacent whenever $v_1v_2 \in E(G)$, or $v_1 = v_2$ and $h_1h_2 \in E(H)$. Additionally, $G * H$ is called a *disjoint product* of G and H if two vertices (v_1, h_1) and (v_2, h_2) are adjacent whenever $v_1 = v_2$ and $h_1h_2 \in E(H)$. For a detailed treatment of graph products, we refer the reader to [4].

More generally, given graphs $G = (V, E)$ and $H_x = (V_x, E_x)$ for every $x \in V$; a generalized product of G and $(H_x)_{x\in V}$, denoted by $G*(H_x)_{x\in V}$, is a graph with the vertex set $\bigcup_{x\in V}(\{x\}\times V_x)$. The graph $G*(H_x)_{x\in V}$ is called a *generalized Cartesian product* of G and $(H_x)_{x\in V}$ if two vertices (x, v_x) and (y, v_y) are adjacent whenever $xy \in E$ and $v_x = v_y$, or $x = y$ and $v_xv_y \in E_x$. The graph $G * (H_x)_{x\in V}$ is called a *generalized direct product* of G and $(H_x)_{x\in V}$ if two vertices (x, v_x) and (y, v_y) are adjacent whenever $xy \in E$ and $v_xv_y \in E_x$. The graph $G * (H_x)_{x\in V}$ is called a *generalized strong product* of G and $(H_x)_{x\in V}$ if it is a generalized Cartesian or generalized direct product. The graph $G * (H_x)_{x\in V}$ is called a *generalized lexicographic product* of G and $(H_x)_{x\in V}$ if two vertices (x, v_x) and (y, v_y) are adjacent whenever $xy \in E$, or $x = y$ and $v_xv_y \in E_x$. Additionally, $G*(H_x)_{x\in V}$ is called a *generalized disjoint product* of G and $(H_x)_{x\in V}$ if two vertices (x, v_x) and (y, v_y) are adjacent whenever $x = y$ and $v_xv_y \in E_x$. Evidently, if $H_x = H$ for any vertex $x \in V$, then the resulting graph is the product $G * H$ of two graphs G and H. In order to properly study graph products, we need some definitions that consider the set product of sets A and B. In particular, if $S \subseteq A \times B$, we define $\pi_1(S) = \{a : (a, b) \in S \text{ where } b \in B\}$. For $s \in \pi_1(S)$, we define $\pi_s(S) = \{b : (s, b) \in S\}$.

Let $\mathcal{I}$ denote the class of all finite simple graphs. For a subclass $\mathcal{P}$ of $\mathcal{I}$, $\mathcal{P}$ is said to be *isomorphism-closed* if $H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and G is isomorphic to H. A *(graphical) property* means a nonempty isomorphism-closed subclass of $\mathcal{I}$. We also say that a graph G has the property $\mathcal{P}$ if $G \in \mathcal{P}$. A nonempty set S of vertices of a graph G is said to be a $\mathcal{P}$*-set* of G if $G[S] \in \mathcal{P}$. For a given property $\mathcal{U}$, a property $\mathcal{P}$ is said to *appear* in $\mathcal{U}$, whenever there is a $\mathcal{P}$-set of G for each $G \in \mathcal{U}$. For properties $\mathcal{U}_1$ and $\mathcal{U}_2$, we define the property $\mathcal{U}_1 * \mathcal{U}_2$ to be the set $\{G * H : G \in \mathcal{U}_1$ and $H \in \mathcal{U}_2\}$ when we refer $*$ as a usual product, and the set $\{G * (H_x)_{x\in V} : G = (V, E) \in \mathcal{U}_1$ and $H_x = (V_x, E_x) \in \mathcal{U}_2$ for all $x \in V\}$ when we refer $*$ as a generalized product. For a survey of properties, we refer the reader to [2].

Given a property $\mathcal{U}$ and a property $\mathcal{P}$ appearing in $\mathcal{U}$; for a graph $G \in \mathcal{U}$, the maximum cardinality of a $\mathcal{P}$-set in G is called the $M_{\mathcal{P}}$*-number* of G and denoted by $M_{\mathcal{P}}(G)$ while the minimum cardinality of a $\mathcal{P}$-set in G is called the $m_{\mathcal{P}}$*-number* of G and denoted by $m_{\mathcal{P}}(G)$. If S is a $\mathcal{P}$-set of a graph G such that $|S| = M_{\mathcal{P}}(G)$ or $|S| = m_{\mathcal{P}}(G)$, we say that S is an $M_{\mathcal{P}}$*-set* or an $m_{\mathcal{P}}$*-set* of G, respectively. Given a property $\mathcal{U}$, a property $\mathcal{Q}$ appearing in $\mathcal{U}$ and a property $\mathcal{P}$ appearing in $\mathcal{U} * \mathcal{U}$; $\mathcal{P}$ is said to be

(i) *left composed* by Q if it satisfies:
for any $G, H \in \mathcal{U}$ and a nonempty subset S of $V(G * H)$, we have

S is a $\mathcal{P}$-set of $G * H$ if and only if $\pi_1(S)$ is a $\mathcal{Q}$-set of G.

(ii) *right composed* by Q if it satisfies:
for any $G, H \in \mathcal{U}$ and a nonempty subset S of $V(G * H)$, we have

S is a $\mathcal{P}$-set of $G * H$ if and only if $\pi_s(S)$ is a $\mathcal{Q}$-set of H for every $s \in \pi_1(S)$.

(iii) *composed* by Q if it is left and right composed by Q.

(iv) *left generalized composed* by Q if it satisfies:
for any $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$ and a nonempty subset S of $V(G * (H_x)_{x\in V})$, we have

S is a $\mathcal{P}$-set of $G * (H_x)_{x\in V}$ if and only if $\pi_1(S)$ is a $\mathcal{Q}$-set of G.

(v) *right generalized composed* by Q if it satisfies:
for any $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$ and a nonempty subset S of $V(G * (H_x)_{x\in V}))$, we have

S is a $\mathcal{P}$-set of $G * (H_x)_{x\in V}$ if and only if $\pi_s(S)$ is a $\mathcal{Q}$-set of H_s for every $s \in \pi_1(S)$.

(vi) *generalized composed* by Q if it is left and right generalized composed by Q.

Obviously, every i^{th} composed property is an i^{th} generalized composed property for $i = 1, 2, 3$. We list below some examples of the generalized composed properties.

- If $*$ is a generalized lexicographic product, $\mathcal{U} = \mathcal{I}$ and $\mathcal{P} = \mathcal{Q} = \{G \in \mathcal{I} : G$ is connected$\}$, then $\mathcal{P}$ is a left generalized composed by Q property.

- If $*$ is a generalized disjoint product with a fixed positive integer r, $\mathcal{U} = \mathcal{I}$ and $\mathcal{P} = \mathcal{Q} = \{G \in \mathcal{I} : G$ is an r-regular graph$\}$, then $\mathcal{P}$ is a right generalized composed by Q property.

- If $*$ is a generalized lexicographic product, $\mathcal{U} = \mathcal{I}$ and $\mathcal{P} = \mathcal{Q} = \{G \in \mathcal{I} : G$ is acyclic$\}$, then $\mathcal{P}$ is a generalized composed by Q property.

- If $*$ a is generalized lexicographic product, $\mathcal{U} = \mathcal{I}$ and $\mathcal{P} = \mathcal{Q} = \{G \in \mathcal{I} : G$ is empty$\}$, then $\mathcal{P}$ is a generalized composed by Q property.

- If $*$ a is generalized lexicographic product, $\mathcal{U} = \mathcal{I}$ and $\mathcal{P} = \mathcal{Q} = \{G \in \mathcal{I} : G$ is complete$\}$, then $\mathcal{P}$ is a generalized composed by Q property.

Some composed and generalized composed properties have been discovered. In 1977, Ravindra and Parthasarathy [8] found that $\{G \in \mathcal{I} : G$ is perfect$\}$ is a composed property for a lexicographic product. In 1978, Bollobás [1] generalized the result of Mândrescu and showed that $\{G \in \mathcal{I} : G$ is c-perfect$\}$ is a generalized composed property for a generalized lexicographic product. In 1991, Mândrescu [6] showed that $\{G \in \mathcal{I} : G$ is c-perfect$\}$ is a composed property for a Cartesian product. From these three examples of discovering property $\mathcal{P}$, we do still not know about the expression of $M_{\mathcal{P}}$ of such graph products. However, in 1975, Geller and Stahl [3] obtained a graph parameter called the independence number $\alpha(G * H)$ of a lexicographic product $G * H$ as follows $\alpha(G * H) = \alpha(G)\alpha(H)$, i.e., $M_{\mathcal{P}}(G * H) = M_{\mathcal{P}}(G)M_{\mathcal{P}}(H)$ where $\mathcal{P} = \{G \in \mathcal{I} : G$ is empty$\}$. Furthermore, it is easy to show that this product $\mathcal{P}$ is a composed property. This motivates us to find $M_{\mathcal{P}}$ and $m_{\mathcal{P}}$ in each type of property $\mathcal{P}$. Determining graph parameter of a graph product in terms of its factors is well studied in graph theory. In this paper, we continue the study of the $M_{\mathcal{P}}$ of generalized product of graphs having a generalized composed property. Namely, Sections 2, 3 and 4 provide results regarding the $M_{\mathcal{P}}$-number of generalized product of graphs where $\mathcal{P}$ is left, right and generalized composed properties, respectively. Finally, Section 5 gives concluding remarks on the $m_{\mathcal{P}}$, gives some applications to specific graph products, properties and parameters of results in Sections 2, 3 and 4 and thanks to our various funding.

2. Left Generalized Composed Properties

In this section, we begin with the $M_{\mathcal{P}}$-number of generalized product of graphs where $\mathcal{P}$ is a left generalized composed property.

Theorem 2.1. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is left generalized composed by Q. Further, let $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$. We have*

$$M_{\mathcal{P}}(G * (H_x)_{x\in V}) = \max\left\{\sum_{x\in S}|V(H_x)| : S \text{ is a } \mathcal{Q}\text{-set of } G\right\}.$$

Proof. We first show that $M_{\mathcal{P}}(G * (H_x)_{x\in V}) \le \max\big\{\sum_{x\in S}|V(H_x)| : S$ is a $\mathcal{Q}$-set of $G\big\}$. Let L be an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$. By the definition of $\mathcal{P}$, $\pi_1(L)$ is a $\mathcal{Q}$-set of G. Thus $M_{\mathcal{P}}(G * (H_x)_{x\in V}) = |L| = \left|\bigcup_{x\in\pi_1(L)}(\{x\}\times\pi_x(L))\right| = \sum_{x\in\pi_1(L)}|\{x\}\times\pi_x(L)| = \sum_{x\in\pi_1(L)}|\pi_x(L)| \le \sum_{x\in\pi_1(L)}|V(H_x)| \le \max\big\{\sum_{x\in S}|V(H_x)| : S$ is a $\mathcal{Q}$-set of $G\big\}$.

Now, we show the rest that $M_{\mathcal{P}}(G * (H_x)_{x\in V}) \ge \max\big\{\sum_{x\in S}|V(H_x)| : S$ is a $\mathcal{Q}$-set of $G\big\}$. Let L be a $\mathcal{Q}$-set of G such that $\sum_{x\in L}|V(H_x)| = \max\big\{\sum_{x\in S}|V(H_x)| : S$ is a $\mathcal{Q}$-set of $G\big\}$. Further, let $L' = \bigcup_{x\in L}(\{x\}\times V(H_x))$. We note that $|L'| = \left|\bigcup_{x\in L}(\{x\}\times V(H_x))\right| = \sum_{x\in L}|\{x\}\times V(H_x)| = \sum_{x\in L}|V(H_x)| = \max\big\{\sum_{x\in S}|V(H_x)| : S$ is a $\mathcal{Q}$-set of $G\big\}$. By the definition of $\mathcal{P}$, L' is a $\mathcal{P}$-set of $G * (H_x)_{x\in V}$ since $\pi_1(L') = L$ is a $\mathcal{Q}$-set of G. Therefore, $M_{\mathcal{P}}(G * (H_x)_{x\in V}) \ge \max\big\{\sum_{x\in S}|V(H_x)| : S$ is a $\mathcal{Q}$-set of $G\big\}$ since $M_{\mathcal{P}}(G * (H_x)_{x\in V}) \ge |L'|$.

Hence $M_{\mathcal{P}}(G * (H_x)_{x\in V}) = \max\big\{\sum_{x\in S}|V(H_x)| : S$ is a $\mathcal{Q}$-set of $G\big\}$. □

Corollary 2.1. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is left composed by Q. Further, let $G = (V, E)$,$H_x \in \mathcal{U}$ for all $x \in V$. If $|V(H_x)| = n$ for all $x \in V$ where n is a positive integer, then we have*

$$M_{\mathcal{P}}(G * (H_x)_{x\in V}) = nM_{\mathcal{Q}}(G).$$

The next corollary is a direct application of Corollary 2.1.

Corollary 2.2. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is left composed by Q. Further, let $G,H \in \mathcal{U}$. We have*

$$M_{\mathcal{P}}(G * H) = |V(H)|M_{\mathcal{Q}}(G).$$

The following result shows a necessary condition of a nonempty vertex set of generalized product graphs to be an $M_{\mathcal{P}}$-set.

Theorem 2.2. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is left generalized composed by Q. Further, let $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$ and let S be a nonempty subset of $V(G * (H_x)_{x\in V})$. If S is an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$, then $\pi_s(S) = V(H_s)$ for every $s \in \pi_1(S)$.*

Proof. Let $V_x = V(H_x)$ for every $x \in V$. The proof is by contraposition. Assume that $\pi_a(S) \neq V_a$ for some $a \in \pi_1(S)$. Clearly, $|V_a| > |\pi_a(S)|$. Then $\left|\left(\bigcup_{x\in\pi_1(S)\setminus\{a\}} (\{x\} \times \pi_x(S))\right) \cup (\{a\} \times V_a)\right|$ $= \left|\bigcup_{x\in\pi_1(S)\setminus\{a\}}(\{x\} \times \pi_x(S))\right| + |\{a\} \times V_a| = \left|\bigcup_{x\in\pi_1(S)\setminus\{a\}}(\{x\} \times \pi_x(S))\right| + |V_a| >$ $\left|\left(\bigcup_{x\in\pi_1(S)\setminus\{a\}} (\{x\} \times \pi_x(S))\right)\right| + |\pi_a(S)| = \left|\left(\bigcup_{x\in\pi_1(S)\setminus\{a\}} (\{x\} \times \pi_x(S))\right)\right| + |(\{a\} \times \pi_a(S))| =$ $\left|\bigcup_{x\in\pi_1(S)} (\{x\} \times \pi_x(S))\right| = |S|$.

By the definition of $\mathcal{P}$, $\left(\bigcup_{x\in\pi_1(S)\setminus\{a\}} (\{x\} \times \pi_x(S))\right) \cup (\{a\} \times V_a)$ is a $\mathcal{P}$-set of $G * (H_x)_{x\in V}$. Therefore, S is not an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$. □

Theorem 2.3. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is left generalized composed by Q. Further, let $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$. If $|V(H_x)| = n$ for all $x \in V$ where n is a positive integer, then S is an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$ if and only if the following conditions hold:*

(1) $\pi_1(S)$ *is an $M_{\mathcal{Q}}$-set of G,*
(2) $\pi_s(S) = V(H_s)$ *for every $s \in \pi_1(S)$.*

Proof. By Theorem 2.2, we need to show the rest that $\pi_1(S)$ is an $M_{\mathcal{Q}}$-set of G. By Corollary 2.1, we have $nM_{\mathcal{Q}}(G) = M_{\mathcal{P}}(G * (H_x)_{x\in V}) = |S| = \left|\bigcup_{x\in\pi_1(S)} (\{x\} \times \pi_x(S))\right| = \sum_{x\in\pi_1(S)} |\{x\} \times \pi_x(S)| = \sum_{x\in\pi_1(S)} |\pi_x(S)| = \sum_{x\in\pi_1(S)} |V(H_x)| = \sum_{x\in\pi_1(S)} n = n\sum_{x\in\pi_1(S)} 1 = n|\pi_1(S)|$. Consequently, $M_{\mathcal{Q}}(G) = |\pi_1(S)|$.

For the converse, we assume that (1) and (2) hold. By the definition of $\mathcal{P}$, S is a $\mathcal{P}$-set of $G * (H_x)_{x\in V}$ because $\pi_1(S)$ is a $\mathcal{Q}$-set of G and $\pi_s(S) = V(H_s)$ for every $s \in \pi_1(S)$. We see that $|S| = \left|\bigcup_{x\in\pi_1(S)} (\{x\} \times \pi_x(S))\right| = \sum_{x\in\pi_1(S)} |\{x\} \times \pi_x(S)| = \sum_{x\in\pi_1(S)} |\pi_x(S)| = \sum_{x\in\pi_1(S)} |V(H_x)| = \sum_{x\in\pi_1(S)} n = n\sum_{x\in\pi_1(S)} 1 = n|\pi_1(S)| = nM_{\mathcal{P}}(G)$. Hence S is an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$ by Corollary 2.1. □

Next, we characterize the $M_{\mathcal{P}}$-set of product graphs.

Corollary 2.3. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is left composed by Q. Further, let $G, H \in \mathcal{U}$ and let S be a nonempty subset of $V(G * H)$. Then S is an $M_{\mathcal{P}}$-set of $G * H$ if and only if the following two conditions hold:*

(1) $\pi_1(S)$ *is an $M_{\mathcal{P}}$-set of G,*
(2) $\pi_s(S) = V(H)$ *for every $s \in \pi_1(S)$.*

We not only obtain the $M_{\mathcal{P}}$-number of product graphs, but we can also enumerate the number of $M_{\mathcal{P}}$-sets of product graphs in the term of the number of $M_{\mathcal{Q}}$-sets of its graph factors.

Theorem 2.4. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is left generalized composed by Q. Further, let $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$. If $|V(H_x)| = n$ for all $x \in V$ where n is a positive integer , $\mathfrak{S}$ is the family of $M_{\mathcal{P}}$-sets of $G * H$, $\mathfrak{S}_1$ is the family of $M_{\mathcal{Q}}$-sets of G, then*

$$|\mathfrak{S}| = |\mathfrak{S}_1|.$$

Proof. We construct an $M_{\mathcal{P}}$-set of $G * H$ in 2 steps as follows.
Step 1 : Choose an $M_{\mathcal{Q}}$-set S_1 from $\mathfrak{S}_1$.
Step 2 : For each $x \in S_1$, build the $M_{\mathcal{P}}$-set $\bigcup_{x\in S_1} (\{x\} \times V(H_x)) \in \mathfrak{S}$.
By the multiplication law and by Corollary 2.3, $|\mathfrak{S}| = \mathfrak{S}_1|$. □

Corollary 2.4. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is left composed by Q. Further, let $G, H \in \mathcal{U}$. If $\mathfrak{S}$ is the family of $M_{\mathcal{P}}$-sets of $G * H$ and $\mathfrak{S}_1$ is the family of $M_{\mathcal{Q}}$-sets of G, then*

$$|\mathfrak{S}| = |\mathfrak{S}_1|.$$

3. Right Generalized Composed Properties

In this section, we begin with the $M_{\mathcal{P}}$-number of generalized product of graphs where $\mathcal{P}$ is a right generalized composed property.

Theorem 3.1. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is right generalized composed by Q. Further, let $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$. We have*

$$M_{\mathcal{P}}(G * (H_x)_{x\in V}) = \sum_{x\in V} M_{\mathcal{Q}}(H_x).$$

Proof. We first show that $M_{\mathcal{P}}(G * (H_x)_{x\in V}) \leq \sum_{x\in V} M_{\mathcal{Q}}(H_x)$. Let L be an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$. By the definition of $\mathcal{P}$, $\pi_x(L)$ is a $\mathcal{Q}$-set of H_x for all $x \in \pi_1(L)$. We have $|\pi_x(L)| \leq M_{\mathcal{Q}}(H_x)$ for all $x \in \pi_1(L)$. Thus $M_{\mathcal{P}}(G * (H_x)_{x\in V}) = |L| = \left|\bigcup_{x\in\pi_1(L)} (\{x\} \times \pi_x(L))\right| = \sum_{x\in\pi_1(L)} |\{x\} \times \pi_x(L)| = \sum_{x\in\pi_1(L)} |\pi_x(L)| \leq \sum_{x\in\pi_1(L)} M_{\mathcal{Q}}(H_x) \leq \sum_{x\in V} M_{\mathcal{Q}}(H_x)$.

Now, we show the rest that $M_{\mathcal{P}}(G * (H_x)_{x\in V}) \geq \sum_{x\in V} M_{\mathcal{Q}}(H_x)$. Let L_x be an $M_{\mathcal{Q}}$-set of H_x for each $x \in V$. Further, let $L' = \bigcup_{x\in V} (\{x\} \times L_x)$. We note that $|L'| = \left|\bigcup_{x\in V} (\{x\} \times L_x)\right| = \sum_{x\in V} |\{x\} \times L_x| = \sum_{x\in V} |L_x| = \sum_{x\in V} M_{\mathcal{Q}}(H_x)$. By the definition of $\mathcal{P}$, L' is a $\mathcal{P}$-set of $G * (H_x)_{x\in V}$ since $\pi_x(L') = L_x$ is a $\mathcal{Q}$-set of H_x for all $x \in \pi_1(L') = V$. Therefore, $M_{\mathcal{P}}(G * (H_x)_{x\in V}) \geq |L'| = \sum_{x\in V} M_{\mathcal{Q}}(H_x)$.

Hence $M_{\mathcal{P}}(G * (H_x)_{x\in V}) = \sum_{x\in V} M_{\mathcal{Q}}(H_x)$. □

Corollary 3.1. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is right generalized composed by Q. Further, let $G = (V, E)$, $H_x \in \mathcal{U}$ for all $x \in V$. If $M_{\mathcal{P}}(H_x) = n$ for all $x \in V$ where n is a positive integer, then we have*

$$M_{\mathcal{P}}(G * (H_x)_{x\in V}) = n|V|.$$

The next corollary is a direct application of Corollary 3.1.

Corollary 3.2. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is right composed by Q. Further, let $G, H \in \mathcal{U}$. We have*

$$M_{\mathcal{P}}(G * H) = |V(G)| M_{\mathcal{Q}}(H).$$

The following result shows a sufficient condition of a nonempty vertex set of generalized product graphs to be an $M_{\mathcal{P}}$-set.

Theorem 3.2. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is right generalized composed by Q. Further, let $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$ and let S be a nonempty subset of $V(G * (H_x)_{x \in V})$. Then S is an $M_{\mathcal{P}}$-set of $G * (H_x)_{x \in V}$ if and only if the following conditions hold:*

(1) $\pi_1(S) = V$,
(2) $\pi_s(S)$ *is an $M_{\mathcal{Q}}$-set of H_s for every $s \in \pi_1(S)$.*

Proof. Let $V_x = V(H_x)$ for all $x \in V$.

The proof is by contraposition. Assume that (1) or (2) does not hold.

Case 1: $\pi_1(S) \neq V$.
Let $a \in V \backslash \pi_1(S)$ and L_a be a $\mathcal{Q}$-set of H_a. Then $\left|\big(\{a\} \times L_a\big) \cup \Big(\bigcup_{x \in \pi_1(S)} \big(\{x\} \times \pi_x(S)\big)\Big)\right| = |\big(\{a\} \times L_a\big) \cup S| = |\{a\} \times L_a| + |S| = |L_a| + |S| > |S| = M_{\mathcal{P}}(G * (H_x)_{x \in V})$. Therefore, S is not an $M_{\mathcal{P}}$-set of $G * (H_x)_{x \in V}$ since $\Big(\{a\} \times L_a\Big) \cup \Big(\bigcup_{x \in \pi_1(S)} \big(\{x\} \times \pi_x(S)\big)\Big)$ is a $\mathcal{P}$-set of $G * (H_x)_{x \in V}$.

Case 2: $\pi_a(S)$ is not an $M_{\mathcal{Q}}$-set of H for some $a \in \pi_1(S)$.
Let L_a be an $M_{\mathcal{Q}}$-set of H_a. Clearly, $M_{\mathcal{Q}}(H_a) = |L_a| > |\pi_a(S)|$. Then $\left|\Big(\bigcup_{x \in \pi_1(S) \backslash \{a\}} \big(\{x\} \times \pi_x(S)\big)\Big) \cup \big(\{a\} \times L_a\big)\right| = \left|\bigcup_{x \in \pi_1(S) \backslash \{a\}} (\{x\} \times \pi_x(S))\right| + |\{a\} \times L_a| = \left|\bigcup_{x \in \pi_1(S) \backslash \{a\}} (\{x\} \times \pi_x(S))\right| + |L_a| > \left|\Big(\bigcup_{x \in \pi_1(S) \backslash \{a\}} \big(\{x\} \times \pi_x(S)\big)\Big)\right| + |\pi_a(S)| = \left|\Big(\bigcup_{x \in \pi_1(S) \backslash \{a\}} \big(\{x\} \times \pi_x(S)\big)\Big)\right| + |\,(\{a\} \times \pi_a(S))\,| = \left|\big(\bigcup_{x \in \pi_1(S) \backslash \{a\}} \big(\{x\} \times \pi_x(S)\big)\big) \cup \big(\{a\} \times \pi_a(S)\big)\right| = \left|\bigcup_{x \in \pi_1(S)} \big(\{x\} \times \pi_x(S)\big)\right| = |S|$. By the definition of $\mathcal{P}$, $\Big(\bigcup_{x \in \pi_1(S) \backslash \{a\}} \big(\{x\} \times \pi_x(S)\big)\Big) \cup \big(\{a\} \times L_a\big)$ is a $\mathcal{P}$-set of $G * (H_x)_{x \in V}$. Therefore, S is not an $M_{\mathcal{P}}$-set of $G * (H_x)_{x \in V}$. □

Next, we characterize the $M_{\mathcal{P}}$-set of product graphs.

Corollary 3.3. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is right composed by Q. Further, let $G, H \in \mathcal{U}$ and let S be a nonempty subset of $V(G * H)$. Then S is an $M_{\mathcal{P}}$-set of $G * H$ if and only if the following two conditions hold:*

(1) $\pi_1(S) = V(H)$,
(2) $\pi_s(S)$ *is an $M_{\mathcal{Q}}$-set of H for every $s \in \pi_1(S)$.*

We not only obtain the $M_{\mathcal{P}}$-number of product graphs, but we can also enumerate the number of $M_{\mathcal{P}}$-sets of product graphs in the term of the number of $M_{\mathcal{P}}$-sets of its graph factors.

Theorem 3.3. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is right generalized composed by Q. Further, let $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$. If $\mathfrak{S}$ is the family of $M_{\mathcal{P}}$-sets of $G * (H_x)_{x\in V}$ and $\mathfrak{S}_x$ is the family of $M_{\mathcal{Q}}$-sets of H_x for all $x \in V$, then*

$$|\mathfrak{S}| = \sum_{x\in V} |\mathfrak{S}_x|.$$

Proof. We construct an $M_{\mathcal{P}}$-set of $G * H$ in 2 steps as follows.
Step 1 : For each $x \in V$, choose an $M_{\mathcal{Q}}$-set S_x from $\mathfrak{S}_x$.
Step 2 : Build the $M_{\mathcal{P}}$-set $\{x\} \times S_x \in \mathfrak{S}$.
By the multiplication law and by Corollary 3.3, $|\mathfrak{S}| = \sum_{x\in V} |\mathfrak{S}_x|$. □

Corollary 3.4. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is right composed by Q. Further, let $G, H \in \mathcal{U}$. If $\mathfrak{S}$ is the family of $M_{\mathcal{P}}$-sets of $G * H$ and $\mathfrak{S}'$ is the family of $M_{\mathcal{Q}}$-sets of H, then*

$$|\mathfrak{S}| = |V(G)||\mathfrak{S}'|.$$

4. Generalized Composed Properties

In this section, we begin with the $M_{\mathcal{P}}$-number of generalized product of graphs where $\mathcal{P}$ is a generalized composed property.

Theorem 4.1. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is generalized composed by Q. Further, let $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$. We have*

$$M_{\mathcal{P}}(G * (H_x)_{x\in V}) = \max\left\{\sum_{x\in S} M_{\mathcal{Q}}(H_x) : S \textit{ is a } \mathcal{Q}\textit{-set of } G\right\}.$$

Proof. We first show that $M_{\mathcal{P}}(G * (H_x)_{x\in V}) \leq \max\big\{\sum_{x\in S} M_{\mathcal{Q}}(H_x) : S \text{ is an } \mathcal{Q}\text{-set of } G\big\}$. Let L be an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$. By the definition of $\mathcal{P}$, $\pi_1(L)$ is a $\mathcal{Q}$-set of G and $\pi_x(L)$ is a $\mathcal{Q}$-set of H_x for all $x \in \pi_1(L)$. We have $|\pi_x(L)| \leq M_{\mathcal{Q}}(H_x)$ for all $x \in \pi_1(L)$. Thus $M_{\mathcal{P}}(G * (H_x)_{x\in V}) = |L| = \left|\bigcup_{x\in\pi_1(L)} (\{x\} \times \pi_x(L))\right| = \sum_{x\in\pi_1(L)} |\{x\} \times \pi_x(L)| = \sum_{x\in\pi_1(L)} |\pi_x(L)| \leq \sum_{x\in\pi_1(L)} M_{\mathcal{Q}}(H_x) \leq \max\big\{\sum_{x\in S} M_{\mathcal{Q}}(H_x) : S \text{ is a } \mathcal{Q}\text{-set of } G\big\}$.

Now, we show the rest that $M_{\mathcal{P}}(G * (H_x)_{x\in V}) \geq \max\big\{\sum_{x\in S} M_{\mathcal{Q}}(H_x) : S \text{ is a } \mathcal{Q}\text{-set of } G\big\}$. Let L be a $\mathcal{Q}$-set of G and L_x be an $M_{\mathcal{Q}}$-set of H_x for each $x \in L$ such that $\sum_{x\in L} M_{\mathcal{Q}}(H_x) = \max\Big\{\sum_{x\in S} M_{\mathcal{Q}}(H_x) : S \text{ is a } \mathcal{Q}\text{-set of } G\Big\}$. Further, let $L' = \bigcup_{x\in L} (\{x\} \times L_x)$. We note that $|L'| = \left|\bigcup_{x\in L} (\{x\} \times L_x)\right| = \sum_{x\in L} |\{x\} \times L_x| = \sum_{x\in L} |M_x| = \sum_{x\in L} M_{\mathcal{Q}}(H_x) = \max\big\{\sum_{x\in S} M_{\mathcal{Q}}(H_x) : S \text{ is a } \mathcal{Q}\text{-set of } G\big\}$. By the definition of $\mathcal{P}$, L' is a $\mathcal{P}$-set of $G * (H_x)_{x\in V}$

since $\pi_1(L') = L$ is a $\mathcal{Q}$-set of G and $\pi_x(L') = L_x$ is a $\mathcal{Q}$-set of H_x for all $x \in \pi_1(L')$. Therefore, $M_{\mathcal{P}}(G * (H_x)_{x\in V}) \geq |L'| = \max\Big\{ \sum_{x\in S} M_{\mathcal{Q}}(H_x) : S \text{ is a } \mathcal{Q}\text{-set of } G\Big\}$.

Hence $M_{\mathcal{P}}(G * (H_x)_{x\in V}) = \max\Big\{ \sum_{x\in S} M_{\mathcal{Q}}(H_x) : S \text{ is a } \mathcal{Q}\text{-set of } G\Big\}$. □

Corollary 4.1. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is generalized composed by Q. Further, let $G = (V,E)$, $H_x \in \mathcal{U}$ for all $x \in V$. If $M_{\mathcal{P}}(H_x) = n$ for all $x \in V$ where n is a positive integer, then we have*

$$M_{\mathcal{P}}(G * (H_x)_{x\in V}) = nM_{\mathcal{Q}}(G).$$

The next corollary is a direct application of Corollary 4.1.

Corollary 4.2. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is composed by Q. Further, let $G, H \in \mathcal{U}$. We have*

$$M_{\mathcal{P}}(G * H) = M_{\mathcal{Q}}(G)M_{\mathcal{Q}}(H).$$

The following result shows a necessary condition of a nonempty vertex set of generalized product graphs to be an $M_{\mathcal{P}}$-set.

Theorem 4.2. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is generalized composed by Q. Further, let $G = (V,E)$, $H_x \in \mathcal{U}$ for every $x \in V$ and let S be a nonempty subset of $V(G * (H_x)_{x\in V})$. If S is an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$, then $\pi_s(S)$ is an $M_{\mathcal{Q}}$-set of H_s for every $s \in \pi_1(S)$.*

Proof. The proof is by contraposition. Assume that $\pi_a(S)$ is not an $M_{\mathcal{Q}}$-set of H for some $a \in \pi_1(S)$. Let L_a be an $M_{\mathcal{Q}}$-set of H_a. Clearly, $M_{\mathcal{Q}}(H_a) = |L_a| > |\pi_a(S)|$. Then $\left|\left(\bigcup_{x\in\pi_1(S)\setminus\{a\}} (\{x\} \times \pi_x(S))\right) \cup (\{a\} \times L_a)\right| = \left|\bigcup_{x\in\pi_1(S)\setminus\{a\}}(\{x\} \times \pi_x(S))\right| + |\{a\} \times L_a| = \left|\bigcup_{x\in\pi_1(S)\setminus\{a\}}(\{x\} \times \pi_x(S))\right| + |L_a| > \left|\left(\bigcup_{x\in\pi_1(S)\setminus\{a\}} (\{x\} \times \pi_x(S))\right)\right| + |\pi_a(S)| = \left|\left(\bigcup_{x\in\pi_1(S)\setminus\{a\}} (\{x\} \times \pi_x(S))\right)\right| + |(\{a\} \times \pi_a(S))| = \left|\bigcup_{x\in\pi_1(S)} (\{x\} \times \pi_x(S))\right| = |S|$. By the definition of $\mathcal{P}$, $\left(\bigcup_{x\in\pi_1(S)\setminus\{a\}} (\{x\} \times \pi_x(S))\right) \cup (\{a\} \times L_a)$ is a $\mathcal{P}$-set of $G * (H_x)_{x\in V}$. Therefore, S is not an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$. □

Theorem 4.3. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is generalized composed by Q. Further, let $G = (V,E)$, $H_x \in \mathcal{U}$ for every $x \in V$. If $M_{\mathcal{Q}}(H_x) = n$ for all $x \in V$ where n is a positive integer, then S is an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$ if and only if the following conditions hold:*

(1) *$\pi_1(S)$ is an $M_{\mathcal{Q}}$-set of G,*
(2) *$\pi_s(S)$ is an $M_{\mathcal{Q}}$-set of H_s for every $s \in \pi_1(S)$.*

Proof. By Theorem 4.2, we need to show the rest that $\pi_1(S)$ is an $M_{\mathcal{Q}}$-set of G. By Corollary 4.1, we have $nM_{\mathcal{Q}}(G) = M_{\mathcal{P}}(G * (H_x)_{x\in V}) = |S| = \left|\bigcup_{x\in\pi_1(S)} (\{x\} \times \pi_x(S))\right| = \sum_{x\in\pi_1(S)} |\{x\} \times \pi_x(S)| = \sum_{x\in\pi_1(S)} |\pi_x(S)| = \sum_{x\in\pi_1(S)} n = n\sum_{x\in\pi_1(S)} 1 = n|\pi_1(S)|$. Consequently, $M_{\mathcal{Q}}(G) = |\pi_1(S)|$.

For the converse, we assume that (1) and (2) hold. By the definition of $\mathcal{P}$, S is a $\mathcal{P}$-set of $G * (H_x)_{x\in V}$ because $\pi_1(S) \in \mathcal{Q}$ and $\pi_s(S) \in \mathcal{Q}$ for every $s \in \pi_1(S)$. We see that $|S| = \left|\bigcup_{x\in\pi_1(S)} (\{x\} \times \pi_x(S))\right| = \sum_{x\in\pi_1(S)} |\{x\} \times \pi_x(S)| = \sum_{x\in\pi_1(S)} |\pi_x(S)| = \sum_{x\in\pi_1(S)} n = n\sum_{x\in\pi_1(S)} 1 = n|\pi_1(S)|$. Hence S is an $M_{\mathcal{P}}$-set of $G * (H_x)_{x\in V}$ by Corollary 4.1. □

Next, we characterize the $M_{\mathcal{P}}$-set of product graphs.

Corollary 4.3. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is composed by Q. Further, let $G,H \in \mathcal{U}$ and let S be a nonempty subset of $V(G * H)$. Then S is an $M_{\mathcal{P}}$-set of $G * H$ if and only if the following two conditions hold:*

(1) *$\pi_1(S)$ is an $M_{\mathcal{Q}}$-set of G,*
(2) *$\pi_s(S)$ is an $M_{\mathcal{Q}}$-set of H for every $s \in \pi_1(S)$.*

We not only obtain the $M_{\mathcal{P}}$-number of product graphs, but we can also enumerate the number of $M_{\mathcal{P}}$-sets of product graphs in the term of the number of $M_{\mathcal{Q}}$-sets of its graph factors.

Theorem 4.4. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is generalized composed by Q. Further, let $G = (V,E)$,$H_x \in \mathcal{U}$ for every $x \in V$. If $M_{\mathcal{Q}}(H_x) = n$ for all $x \in V$ where n is a positive integer , $\mathfrak{S}$ is the family of $M_{\mathcal{P}}$-sets of $G * H$, $\mathfrak{S}_1$ is the family of $M_{\mathcal{Q}}$-sets of G and $\mathfrak{S}_x$ is the family of $M_{\mathcal{Q}}$-sets of H_x for all $x \in V$, then*

$$|\mathfrak{S}| = \sum_{S_1\in\mathfrak{S}_1} \prod_{x\in S_1} |\mathfrak{S}_x|.$$

Proof. We construct an $M_{\mathcal{P}}$-set of $G * H$ in 3 steps as follows.
Step 1 : Choose an $M_{\mathcal{Q}}$-set S_1 from $\mathfrak{S}_1$.
Step 2 : For each $x \in S_1$, choose an $M_{\mathcal{Q}}$-set S_x from $\mathfrak{S}_2$.
Step 3 : Build the $M_{\mathcal{P}}$-set $\bigcup_{x\in S_1} (\{x\} \times S_x) \in \mathfrak{S}$.
By the multiplication law and by Corollary 4.3, $|\mathfrak{S}| = \sum_{S_1\in\mathfrak{S}_1} \prod_{x\in S_1} |\mathfrak{S}_x|$. □

Corollary 4.4. *For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$ such that $\mathcal{P}$ is composed by Q. Further, let $G,H \in \mathcal{U}$. If $\mathfrak{S}$ is the family of $M_{\mathcal{P}}$-sets of $G * H$, $\mathfrak{S}_1$ is the family of $M_{\mathcal{Q}}$-sets of G and $\mathfrak{S}_2$ is the family of $M_{\mathcal{Q}}$-sets of H, then*

$$|\mathfrak{S}| = |\mathfrak{S}_1||\mathfrak{S}_2|^{M_{\mathcal{Q}}(G)}.$$

5. Conclusion

For convenience in this section, we denote "composed by" and "generalized composed by" by "CB" and "GCB", respectively. We summarize corresponding results for the $m_{\mathcal{P}}(G * H)$ and the $m_{\mathcal{P}}(G * (H_x)_{x \in V})$ as follows.

For a property $\mathcal{U}$, let $\mathcal{Q}$ be a property appearing in $\mathcal{U}$ and $\mathcal{P}$ a property appearing in $\mathcal{U} * \mathcal{U}$. Let $G = (V, E)$, $H_x \in \mathcal{U}$ for every $x \in V$. We have

$$m_{\mathcal{P}}(G * (H_x)_{x \in V}) = \begin{cases} m_{\mathcal{P}}(G) & \text{, if } \mathcal{P} \text{ is left GCB } Q; \\ \min\left\{\sum_{x \in S} m_{\mathcal{Q}}(H_x) : S \text{ is a subset of } V\right\} & \text{, if } \mathcal{P} \text{ is right GCB } Q; \\ \min\left\{\sum_{x \in S} m_{\mathcal{Q}}(H_x) : S \text{ is a } \mathcal{P}\text{-set of } G\right\} & \text{, if } \mathcal{P} \text{ is GCB } Q. \end{cases}$$

Further, let G,$H \in \mathcal{U}$. We have

$$m_{\mathcal{P}}(G * H) = \begin{cases} m_{\mathcal{P}}(G) & \text{, if } \mathcal{P} \text{ is left CB } Q; \\ m_{\mathcal{Q}}(H) & \text{, if } \mathcal{P} \text{ is right CB } Q; \\ m_{\mathcal{P}}(G) m_{\mathcal{Q}}(H) & \text{, if } \mathcal{P} \text{ is CB } Q. \end{cases}$$

Next, we give two tables showing some specific results for composed and generalized composed properties obtained by applying our results in Sections 2, 3 and 4.

Table 1. Composed properties

$\mathcal{U}$	$\mathcal{P}$	$\mathcal{Q}$	$*$	$M_{\mathcal{P}}(G*H)$	reason
$\mathcal{I}$	$\{G \in I : G$ is connected$\}$	$\mathcal{P}$	lexicographic	$M_{\mathcal{Q}}(G)\lvert V(H)\rvert$	left CB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is r-regular$\}$	$\mathcal{P}$	disjoint	$\lvert V(G)\rvert M_{\mathcal{Q}}(H)$	right CB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is empty$\}$	$\mathcal{P}$	lexicographic	$\alpha(G)\alpha(H)$	[3] or CB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is acyclic$\}$	$\mathcal{P}$	lexicographic	$M_{\mathcal{Q}}(G)M_{\mathcal{Q}}(H)$	CB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is complete$\}$	$\mathcal{P}$	lexicographic	$\omega(G)\omega(H)$	PIS
$\mathcal{I}$	$\{G \in I : G$ is perfect$\}$	$\mathcal{P}$	lexicographic	$M_{\mathcal{Q}}(G)M_{\mathcal{Q}}(H)$	[8] and CB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is c-perfect$\}$	$\mathcal{P}$	lexicographic	$M_{\mathcal{Q}}(G)M_{\mathcal{Q}}(H)$	[1] and CB $\mathcal{Q}$
$\{G \in \mathcal{I} : G$ is nontrivial$\}$	$\{G \in \mathcal{I} : G$ is c-perfect$\}$	$\{G \in \mathcal{I} : G$ is bipartite$\}$	Cartesian	$M_{\mathcal{Q}}(G)M_{\mathcal{Q}}(H)$	[6] and CB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is s-perfect$\}$	$\mathcal{P}$	lexicographic	$M_{\mathcal{Q}}(G)M_{\mathcal{Q}}(H)$	[1] and CB $\mathcal{Q}$

Table 2. Generalized composed properties

$\mathcal{U}$	$\mathcal{P}$	$\mathcal{Q}$	$*$	$M_{\mathcal{P}}(G*(H_x)_{x\in V})$	reason
$\mathcal{I}$	$\{G \in I : G$ is connected$\}$	$\mathcal{Q}$	generalized lexicographic	$\max\{\sum_{x\in S}\lvert V(H_x)\rvert : S$ is a $\mathcal{Q}$-set of $G\}$	left GCB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is r-regular$\}$	$\mathcal{P}$	generalized disjoint	$\sum_{x\in V} M_{\mathcal{Q}}(H_x)$	right GCB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is empty$\}$	$\mathcal{P}$	generalized lexicographic	$\max\{\sum_{x\in S}\alpha(H_x) : S$ is a $\mathcal{Q}$-set of $G\}$	[3] or GCB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is acyclic$\}$	$\mathcal{P}$	generalized lexicographic	$\max\{\sum_{x\in S} M_{\mathcal{Q}}(H_x) : S$ is a $\mathcal{Q}$-set of $G\}$	GCB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is complete$\}$	$\mathcal{P}$	generalized lexicographic	$\max\{\sum_{x\in S}\omega(H_x) : S$ is a $\mathcal{Q}$-set of $G\}$	GCB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is c-perfect$\}$	$\mathcal{P}$	generalized lexicographic	$\max\{\sum_{x\in S} M_{\mathcal{Q}}(H_x) : S$ is a $\mathcal{Q}$-set of $G\}$	[1] and GCB $\mathcal{Q}$
$\mathcal{I}$	$\{G \in I : G$ is s-perfect$\}$	$\mathcal{P}$	generalized lexicographic	$\max\{\sum_{x\in S} M_{\mathcal{Q}}(H_x) : S$ is a $\mathcal{Q}$-set of $G\}$	[1] and GCB $\mathcal{Q}$

Where α and ω denote the independence number and the clique number, respectively.

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