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Orthogonal labeling

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Abstract

Let Δ_G be the maximum degree of a simple connected graph $G(V, E)$. An injective mapping $P : V \rightarrow \mathbb{R}^{\Delta_G}$ is said to be an orthogonal labeling of G if $uv, uw \in E$ implies $(P(v) - P(u)) \cdot (P(w) - P(u)) = 0$, where $\cdot$ is the usual dot product defined in Euclidean space. A graph G which has an orthogonal labeling is called an orthogonal graph. This labeling is motivated by the existence of some labelings defined on some algebraic structure, i.e. harmonious labeling and group distance magic labeling. In this paper we study some preliminary results on orthogonal labeling. One of the early results is the fact that cycles with even number of vertices are orthogonal, while cycles with odd number of vertices are not. The main results in this paper state that any graph containing K_3 as a subgraph is non-orthogonal and that a graph G' obtained from adding a pendant to a vertex in an orthogonal graph G is orthogonal. Moreover, we show that any tree is orthogonal.

Keywords: separate, Euclidean space, orthogonal, orthogonal labeling
Mathematics Subject Classification : 05C78

1. Introduction

A general definition of labeling of a graph is a map that carries some set of graph elements to set of numbers such that a certain condition is fulfilled. For example, magic labeling is a one-to-one map onto the appropriate set of consecutive integers starting from 1, with some kind of "constant-sum" property. We can find other examples on [4]. Moreover, there are some labelings which map to a general set. An example that satisfies this condition is harmonious labeling, which

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maps the set of vertices to $\mathbb{Z}_{|V|}$. Froncek in [3] and Cichacz in [2] worked on group distance magic labeling which maps vertices to a finite group.

Inspired by these examples, we define a labeling that maps the set of vertices to Euclidean space $\mathbb{R}^n$ and call this labeling: orthogonal labeling. A graph G that has an orthogonal labeling is called an orthogonal graph. This labeling has some similarities with the $n-$cube graph which we show latter that it is indeed an orthogonal graph. Note that orthogonal labeling is also used in different meaning such as in connection with orthogonal double covers. See for example [5]. The main purpose of this paper is to introduce this labeling and some preliminaries results.

2. Orthogonal Labeling

Let Δ_G be the maximum degree of a simple connected graph $G(V, E)$. An orthogonal labeling is a mapping $P : V \to \mathbb{R}^{\Delta_G}$ such that if $uv, uw \in E$, then the images of u, v, w, which are $P(u), P(v), P(w)$, are vectors in $\mathbb{R}^{\Delta_G}$ such that the vector $P(v) - P(u)$ is orthogonal to $P(w) - P(u)$. This leads us to a more formal definition.

Definition 1. *Let $G(V, E)$ be a simple connected graph. An injective mapping $P : V \to \mathbb{R}^{\Delta_G}$ is said to be an orthogonal labeling of G if $uv, uw \in E$ implies $(P(v) - P(u)) \cdot (P(w) - P(u)) = 0$, where $\cdot$ is the usual dot product defined in the Euclidean space $\mathbb{R}^{\Delta_G}$. A graph G which has an orthogonal labeling is said to be orthogonal. A non-orthogonal graph is a graph that does not have an orthogonal labeling.*

For the sake of convenience, we say that a graph that consists of a single vertex is orthogonal.

It is obvious that the graph C_4 is orthogonal. Let $V(C_4) = \{v_1, v_2, v_3, v_4\}$ and $E(C_4) = \{v_iv_{i+1} : i = 1, 2, 3, 4\}$, where v_5 is defined to be v_1. Since $\Delta_{C_4} = 2$, then the injective mapping $P : V(C_4) \to \mathbb{R}^2$ defined by $P(v_1) = (0, 0), P(v_2) = (1, 0), P(v_3) = (1, 1)$, and $P(v_4) = (0, 1)$ is obviously an orthogonal labeling of C_4.

Since orthogonality with respect to usual dot product in Euclidean space is the same as perpendicularity whenever the space is $\mathbb{R}^n,\ n = 1, 2, 3$, then we can induce an orthogonal labeling from drawing the graph in $\mathbb{R}^n$ such that each pair of edges that incident to a same vertex is perpendicular to each other. Thus, by drawing C_4 as a square in $\mathbb{R}^2$, it is obvious that C_4 is orthogonal.

For simplicity we will use P_v to denote $P(v)$.

3. Some Results on Orthogonal Graphs

The graph C_4 can also be seen as a special case of n-cube graph Q_n, in this case $n = 2$. The hypercube or n-cube Q_n is defined as K_2 if $n = 1$, and

$$Q_n = Q_{n-1} \times K_2 \quad \text{if } n \geq 2.$$

It can be easily seen that Q_n is a regular graph of degree n and has 2^n vertices.

The cubes $Q_1,\ Q_2$, and Q_3 can simply be drawn as a line segment, a square, and a cube [1]. From earlier discussion we know that those drawings show that $Q_1,\ Q_2$, and Q_3 are orthogonal. The cube Q_4 is shown in Figure 1.

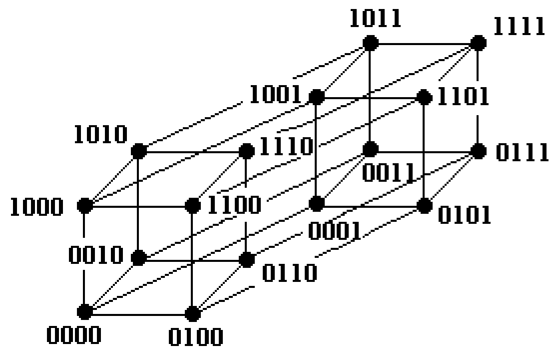

Figure 1. 4-cube, Q_4

A better way to describe the n-cube is to represent its vertex set by the collection of all n-tuples, where each coordinate of the n-tuple is 0 or 1, and where two vertices of Q_n are adjacent if and only if corresponding n-tuples differ in exactly one coordinate. Figure 1 above shows the labeling of the vertices of Q_n for $n = 4$. The way of representing the vertices of Q_n as n-tuples as described above induces an orthogonal labeling. We argue that it is indeed an orthogonal labeling.

Since each vertex in Q_n has a unique representation in n-tuples, we know that this mapping is injective. Let $u, v, w \in V(Q_n)$ such that $uv, uw \in E(Q_n)$ and let p_u, p_v, p_w be the n-tuple representations of u, v, w respectively. Since two vertices of Q_n are adjacent if and only if corresponding n-tuples differ in exactly one coordinate, then $(p_v - p_u)$, $(p_w - p_u)$ must be unit vectors that have value 1 or -1 in exactly one coordinate. This implies $(p_v - p_u) \cdot (p_w - p_u) = 0$. Thus we have proved the following theorem.

Theorem 3.1. *The n-cube graph Q_n is orthogonal for all $n \in \mathbb{N}$.*

We have shown that C_4 is orthogonal.In general, C_{2k} is orthogonal for any natural number $k \geq 2$.

Theorem 3.2. *C_{2k} is orthogonal, $k \in \mathbb{N}, k \geq 2$.*

Proof. Suppose that $V(C_{2k}) = \{v_i : i = 0, 1, 2, \ldots, 2k-1\}$ and $E(C_{2k}) = \{v_i v_{i+1} : i = 0, 1, 2, \ldots, 2k-1\}$, where $v_{2k} = v_0$. We know that $\Delta_{C_{2k}} = 2$. Define a mapping $P : V(C_{2k}) \to \mathbb{R}^2$ as follows:

$$P(v_{2r+1}) = p_{2r+1} = (r, k-r-1), \text{for } r = 0, 1, 2, \ldots, k-1$$

$$P(v_{2r}) = p_{2r} = (r,\ k-r),\ \text{for } r = 1,\ 2, \ldots,\ k-1$$

$$p_0 = (0,\ 0)$$

Notice that the first coordinates of p_{2i} and p_{2j+1} are equal if and only if $i = j$. Despite the equality of first coordinates of p_{2r} and p_{2r+1}, we can see that the odd-even parity of the second coordinates must be different or differ by k. Since the first coordinates of p_{2r} and p_{2r+1} are strictly increasing functions of r, we can deduce that P is an injective function.

We see that

$$(p_{2k-1} - p_0) \cdot (p_1 - p_0) = p_{2k-1} \cdot p_1 = (k-1, 0) \cdot (0, k-1) = 0,$$

$$(p_0 - p_1) \cdot (p_2 - p_1) = (0, -k+1) \cdot (1, 0) = 0,$$

$$(p_0 - p_{2k-1}) \cdot (p_{2k-2} - p_{2k-1}) = (-k+1, 0) \cdot (0, 1) = 0,$$

$$(p_{2r+1} - p_{2r}) \cdot (p_{2(r-1)+1} - p_{2r}) = (0, -1) \cdot (1, 0) = 0, \text{for } r = 1, 2, \ldots, k-1,$$

$$\text{and, } (p_{2(r+1)} - p_{2r+1}) \cdot (p_{2r} - p_{2r+1}) = (1, 0) \cdot (0, 1) = 0, \text{for } r = 2, 3, \ldots, k-2$$

Since all the dot products are zero, we conclude that P is indeed an orthogonal labeling on C_{2k}. Therefore C_{2k} is orthogonal. □

Intuitively we know that C_{2k+1} is non-orthogonal for all natural number k. It is true and we can show it using the fact that a simple connected graph with $\Delta \leq 3$ is orthogonal if and only if it can be drawn such that all edges sharing an endpoint are pairwise perpendicular in $\mathbb{R}^{\Delta}$.

Theorem 3.3. *C_{2k+1} is non-orthogonal for all natural number k.*

Proof. Suppose on the contrary that C_{2k+1} is orthogonal, then we can draw it on $\mathbb{R}^2$ such that all edges sharing an endpoints are pairwise perpendicular. Without loss of generality assume that the first edge is drawn horizontally. In order to maintain the perpendicularity we need to have the second edge and the $(2k+1)$st edge to be drawn vertically. Since the second edge is drawn vertically, we need the third edge to be drawn horizontally. Continuing this argument will lead us to a conclusion that $(2r+1)$st edge must be drawn horizontally for $r = 0,\ 1,\ 2,\ \ldots,\ k$. But this contradicts the fact that the $(2k+1)$st edge is drawn vertically. Therefore C_{2k+1} is non-orthogonal. □

On the theory of planar graphs, we know that if a graph G has a subgraph that is not planar then neither is G. However, if a simple connected graph G has a non-orthogonal simple connected subgraph G', then it is not necessarily true that G must be orthogonal.

Let G be the graph C_5 with one additional vertex v_0 that is adjacent to exactly one vertex of C_5. We have $V(G) = \{v_i : i = 0,\ 1,\ \ldots,\ 5\}$ and $E(G) = \{v_i v_{i+1} : i = 0,\ 1,\ \ldots,\ 5\}$, where $v_6 = v_1$. It is clear that $\Delta_G = 3$. Defined $P : V(G) \to \mathbb{R}^3$ by $p_0 = (0,\ 0,\ 1)$, $p_1 = (0,\ 0,\ 0)$, $p_2 = (1,\ 0,\ 0)$, $p_3 = (1,\ 1,\ 0)$, $p_4 = (1,\ 1,\ 1)$, and $p_5 = (0,\ 0,\ 1)$, where $P(v_i) = p_i,\ i = 0,\ 1,\ \ldots,\ 5$. It is easily seen that P is injective. Notice that

$$(p_0 - p_1) \cdot (p_2 - p_1) = (0,\ 0,\ 1) \cdot (1,\ 0\ ,\ 0) = 0,$$

$$(p_0 - p_1) \cdot (p_5 - p_1) = (0,\ 0,\ 1) \cdot (0,\ 0\ ,\ 1) = 0,$$

$$(p_2 - p_1) \cdot (p_5 - p_1) = (0,\ 1,\ 0) \cdot (0,\ 0\ ,\ 1) = 0,$$

$$(p_1 - p_2) \cdot (p_3 - p_2) = (-1,\ 0,\ 0) \cdot (0,\ 1,\ 0) = 0,$$

$$(p_4 - p_3) \cdot (p_2 - p_3) = (0,\ 0,\ 1) \cdot (0,\ -1,\ 0) = 0,$$

$$(p_5 - p_4) \cdot (p_3 - p_4) = (-1,\ -1,\ 0) \cdot (0,\ 0,\ -1) = 0, \text{ and}$$
$$(p_1 - p_5) \cdot (p_4 - p_5) = (0,\ 0,\ -1) \cdot (1,\ 1,\ 0) = 0.$$

Thus we can conclude that P is an orthogonal labeling of G. We can generally prove this result if the C_5 is substituted by $C_n,\ n > 3$ in a similar fashion.

Theorem 3.4. *Let G be a graph obtained from adding a pendant to C_n. Then G is orthogonal.*

Proof. Let $V(G) = \{v_i : i = 0, 1, \ldots, n\}$ and $E(G) = \{v_i v_{i+1} : i = 0,\ 1,\ \ldots, n\}$, where $v_{n+1} = v_1$. It is obvious that $\Delta_G = 3$. Define a mapping $P : V(G) \to \mathbb{R}^3$ by

$$p_0 = (0, 0, 1),\ p_1 = (0, 0, 0),$$
$$p_{2k} = p_{2k-1} + (1, 0, 0), 1 \leq k \leq \frac{n-1}{2},$$
$$p_{2k+1} = p_{2k} + (0, 1, 0), 1 \leq k \leq \frac{n-1}{2}, \text{ and}$$
$$p_n = p_{n-1} + (0, 0, 1),$$

where $p_i = P(v_i),\ i = 0,\ 1,\ \ldots,\ n$. It can easily be seen that v_0 is the only vertex that is mapped to $(0,\ 0,\ 1)$ and that p_i is an increasing function in i for $i \in \{1,\ 2,\ \ldots n\}$, hence P is injective. We can easily check that

$$(p_{i+1} - p_i) \cdot (p_{i-1} - p_i) = 0, \text{ for } i = 1,\ 2,\ \ldots,\ n.$$

Thus P is an orthogonal labeling on G. $\square$

We will now prove some simple results but truly are useful to determine if a graph is non-orthogonal.

Theorem 3.5. *Let G be a simple connected graph and G' be a connected subgraph of G, such that $\Delta_{G'} = \Delta_G = \Delta$. If G is orthogonal, then so is G'.*

Proof. Let $P : V(G) \to \mathbb{R}^\Delta$ be the orthogonal labeling of G. Since $\Delta_{G'} = \Delta$, we can defined a labeling for $V(G')$ by restricting P to $V(G')$. Since P is injective, so is its restriction. Suppose $u, v, w \in V(G')$ such that $uv,\ uw \in E(G')$. Since $E(G') \subseteq E(G)$, we have

$$(P_{|V(G')}(v) - P_{|V(G')}(u)) \cdot (P_{|V(G')}(w) - P_{|V(G')}(u)) =$$
$$(P(v) - P(u)) \cdot (P(w) - P(u)) = 0.$$

Thus proving that $P_{|V(G')}$ is an orthogonal labeling of G'. $\square$

The contrapositive of Theorem 3.5 is somewhat more useful than the statement of the theorem itself.

Corollary 3.1. *Let G be a simple connected graph and G' be a connected subgraph of G, such that $\Delta_{G'} = \Delta_G$. If G' is non-orthogonal, then so is G.*

We now prove a stronger result.

Theorem 3.6. *Let G be a simple connected graph. If K_3 is a subgraph of G, then G is non-orthogonal.*

Proof. Suppose on the contrary G is orthogonal, then there exists an orthogonal labeling P of G. Let $v_i, i = 1, 2, 3$ be the vertices of the subgraph K_3 of G and $p_i = P(v_i), i = 1, 2, 3$.We will have

$$(p_2 - p_1) \cdot (p_3 - p_1) = 0 \text{ and}$$

$$(p_1 - p_2) \cdot (p_3 - p_2) = 0 \Rightarrow (p_2 - p_1) \cdot (p_2 - p_3) = 0.$$

If we add these equations, we will have

$$(p_2 - p_1) \cdot (p_2 - p_1) = 0 \Rightarrow p_2 - p_1 = 0 \Rightarrow p_2 = p_1.$$

This contradicts the fact that P is an orthogonal labeling, that is an injective function. Therefore G must be non-orthogonal. □

Theorem 3.6 works in a way such that many classes of graph can be determined to be non-orthogonal. We state some of those classes below.

Corollary 3.2. *The graphs K_n and W_n are non-orthogonal for $n > 2$.*

We also have another corollary.

Corollary 3.3. *If G is an orthogonal simple graph with cycle, then the girth of G is at least 4.*

We have seen some properties that ensure the non-orthogonality of some classes of graph. Thus we proceed to a strong lemma concerning orthogonal graph construction. This lemma is a general result of Theorem 3.4.

Lemma 3.1. *Let G be an orthogonal graph. If G' is obtained from adding a pendant to a vertex in G , then G' is orthogonal.*

Proof. Since G' is obtain from adding a pendant to G so that the addition vertex is adjacent to exactly one vertex in G, then $\Delta_{G'} = \Delta_G$ or $\Delta_{G'} = \Delta_G + 1$. Let $P : V(G) \to \mathbb{R}^{\Delta_G}$ be the orthogonal labeling on G. We will define a mapping $P' : V(G') \to \mathbb{R}^{\Delta_{G'}}$ so that P' is an orthogonal labeling on G'. First we will define P' on $V(G) \subseteq V(G')$. If $\Delta_{G'} = \Delta_G$, we let $P'(v) = P(v),\ v \in V(G)$. If $\Delta_{G'} = \Delta_G + 1$, we let $P'(v) = (P(v), 0), v \in V(G)$.

Let u be the vertex that is adjacent to the pendant vertex u'. It is clear that $\delta_{G'}(u) \leq \Delta_{G'}$. Let $\{v_1, v_2, \ldots, v_{\delta_{G'}(u)-1}\}$ be the set of vertices to which u is adjacent to in G. Let $p_i = P'(v_i), i = 1,\ 2,\ \ldots, \delta_{G'}(u) - 1$ and $p_0 = P'(u)$. It suffices to assign the value of $P'(u')$ satisfying $(P'(u') - p_0) \cdot (p_i - p_0) = 0,$ for $i = 1,\ 2,\ \ldots,\ \delta_{G'}(u) - 1$, so that P' is an orthogonal mapping on G'.

Since P is an orthogonal mapping on G, we have $S = \{p_i - p_0 : i = 1,\ 2,\ \ldots,\ \delta_{G'}(u) - 1\}$ be a set of orthogonal vectors in $\mathbb{R}^{\Delta_{G'}}$. In view of Gram-Schmidt theorem we can have a vector p', such that $S \cup \{p'\}$ is orthogonal. However, $S \cup \{tp'\}$ is still orthogonal for any non-zero real number t. Since $P'(V(G'))$ is finite, we can choose a non-zero real number t such that $tp'' + p_0 \notin P'(V(G'))$. In view of our earlier discussion above we conclude that if $P'(u')$is defined to be $tp' + p_0$, we will have P' be an orthogonal labeling on G'. □

Since hairy-cycle graphs can be obtained from adding pendants, as described in Lemma 3.10, to graph G which is described in Theorem 3.4, we have this following theorem.

Theorem 3.7. *Any hairy-cycle graph of $C_n, n > 3$ is orthogonal.*

Further, we see that Lemma 3.1 is indeed a strong lemma. Since any tree T has at least two vertices of degree one, we can remove one of these vertices and the edge incident to it and have a smaller tree. Continuing this process will yield to a tree of the form K_2. Retracing this process gives us an algorithm to construct T from K_2 by adding pendants successively. In view of this argument and repeated application of Lemma 3.1, we have just proved Theorem 3.8.

Theorem 3.8. *Let G be an orthogonal graph and T be a tree. If G' is obtained from attaching the vertex v_0 of T, where $\delta_T(v_0) = 1$, to G so that the v_0 is adjacent to exactly one vertex in G, then G' is orthogonal.*

Let G be an orthogonal graph and T be a tree such that $|V(G) \cap V(T)| = 1$. Consider the set $V(G) \cap V(T) = \{v_0\}$ and $N_T(v_0) = \{v_1, v_2, ..., v_d\}$, where $N_T(v_0)$ is the neighborhood of v_0 in T and $d = \delta_T(v_0)$. Let T_i be the largest subgraph of T containing v_i but not containing v_0. It is easy to see that $\delta_{T_i}(v_i) = 1$. By attaching $v_1 \in V(T_1)$ to G as described in Theorem 3.8, we have another orthogonal graph. Attaching $v_2 \in V(T_2)$ to the resulting graph yield us another orthogonal graph. By continuing the process we will have the resulting orthogonal graph be $G \cup T$. Thus we have proved Theorem 3.9

Theorem 3.9. *If G is an orthogonal graph and T is a tree such that $|V(G) \cap V(T)| = 1$, then $G \cup T$ is orthogonal.*

We conclude our discussion by stating a direct implication of Theorem 3.8.

Theorem 3.10. *Any tree is orthogonal.*

4. Conclusion

In this paper we have some main results stating that any graph containing K_3 as its subgraph is non-orthogonal and that a graph G' obtained from adding a pendant to a vertex in orthogonal graph G is orthogonal. As a corollary of the latter we have that any tree is orthogonal. For further study we can generalize the result for orthogonal labeling on inner product space F^n, where F is a field. However, we need to make sure that the inner product to be consistent in F^n for any natural number n.

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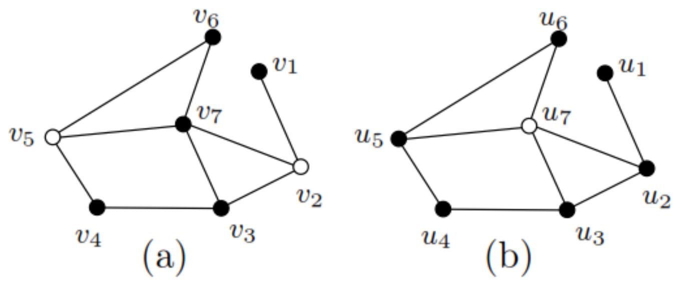

Figure 2. This is the caption of figure

Table 1. This is the table caption

Graph	Notation	Lower Bound	Upper Bound
Cycle	C_n	$n+2$	n^2+2
Path	P_n	$n+1$	n^2+1
Complete graph	K_n	2	$n+4$
Complete bipartite graph	$K_{m,n}$	$2n+2$	$3n^2+2$

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