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On r-dynamic coloring of some graph operations

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Abstract

Let G be a simple, connected and undirected graph. Given r, k as any natural numbers. By an r-dynamic k-coloring of graph G, we mean a *proper* k-coloring $c(v)$ of G such that $|c(N(v))| \geq min\{r, d(v)\}$ for each vertex v in $V(G)$, where $N(v)$ is the neighborhood of v. The r-dynamic chromatic number, written as $\chi_r(G)$, is the minimum k such that G has an r-dynamic k-coloring. We note that the 1-dynamic chromatic number of graph is equal to its chromatic number, denoted by $\chi(G)$, and the 2-dynamic chromatic number of graph has been studied under the name a dynamic chromatic number, denoted by $\chi_d(G)$. By simple observation, we can show that $\chi_r(G) \leq \chi_{r+1}(G)$, however $\chi_{r+1}(G) - \chi_r(G)$ can be arbitrarily large, for example $\chi(Petersen) = 2, \chi_d(Petersen) = 3$, but $\chi_3(Petersen) = 10$. Thus, finding an exact values of $\chi_r(G)$ is not trivially easy. This paper will describe some exact values of $\chi_r(G)$ when G is an operation of special graphs.

Keywords: r-dynamic coloring, r-dynamic chromatic number, graph operations
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1. Introduction

We refer all basic definition of graph to a handbook of graph theory written by Gross *et. al* [1]. Let $G = (V, E)$ be a simple, connected and undirected graph with vertex set V and edge set E, and $d(v)$ be a degree of any $v \in V(G)$. The maximum degree and the minimum degree of G

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are denoted by $\Delta(G)$ and $\delta(G)$, respectively. By a proper k-coloring of a graph G, we mean a map $c : V(G) \rightarrow S$, where $|S| = k$, such that any two adjacent vertices receive different colors. An r-dynamic k-coloring is a proper k-coloring c of G such that $|c(N(v))| \geq min\{r, d(v)\}$ for each vertex v in $V(G)$, where $N(v)$ is the neighborhood of v and $c(S) = \{c(v) : v \in S\}$ for a vertex subset S. The r-dynamic chromatic number, written as $\chi_r(G)$, is the minimum k such that G has an r-dynamic k-coloring. Note that the 1-dynamic chromatic number of graph is equal to its chromatic number, denoted by $\chi(G)$, and the 2-dynamic chromatic number was introduced by Montgomery [5] under the name a dynamic chromatic number, denoted by $\chi_d(G)$. He conjectured $\chi_2(G) \leq \chi(G)+2$ when G is regular, which remains open. Akbari *et. al* [4] proved Montgomery's conjecture for bipartite regular graphs. Lai, *et.al* [6] proved $\chi_2(G) \leq \Delta(G) + 1$ when $\Delta(G) \geq 3$ and no component contains C_5. Kim *et. al* [3] proved $\chi_2(G) \leq 4$ when G is planar and no component is C_5 and also $\chi_d \leq 5$ whenever G is planar.

Obviously, $\chi(G) \leq \chi_2(G)$, but it was shown in [6] that the difference between the chromatic number and the dynamic chromatic number can be arbitrarily large. However, it was conjectured that for regular graphs the difference is at most 2. Some properties of dynamic coloring were studied in [3, 4, 6]. It was proved in [8] that, for a connected graph G, if $\Delta(G) \leq 3$, then $\chi_2(G) \leq 4$ unless $G = C_5$, in which case $\chi_2(C_5) = 5$ and if $\Delta(G) \geq 4$ then $\chi(G) \leq \Delta + 1$. Considering those results, finding an exact value of $\chi_r(G)$ is significantly useful as there are a little number of results provide an exact value of $\chi_r(G)$. Thus, in this paper we will show it when G is an operation of special graphs.

Some Useful Theorem

The following Theorem are useful for determining the dynamic coloring of graphs. Jahanbekam *et. al* [7] characterize the upper bound of $\chi_r(G)$ in term of the diameter of graph.

Theorem 1.1. *[7] If $diam(G) = 2$, then $\chi_2(G) \leq \chi(G) + 2$, with equality holds only when G is a complete bipartite graph or C_5.*

Theorem 1.2. *[7] If G is a k-chromatic graph with diameter at most 3, then $\chi_2(G) \leq 3k$, and this bound is sharp when $k \geq 2$.*

In term of the maximum degree of graph, the r-dynamic of graph satisfies as follows

Observation 1. *[7] $\chi_r(G) \geq min\{\Delta(G), r\} + 1$, and this is sharp. If $\Delta(G) \leq r$ then $\chi_r(G) = min\{\Delta(G), r\}$.*

Theorem 1.3. *[7] $\chi_r(G) \leq r\Delta(G) + 1$, with equality for $r \geq 2$ if and only if G is r-regular with diameter 2 and girth 5.*

The last for the graph operations, Jahanbekam *et. al* proved the following theorem.

Theorem 1.4. *[7] If $\delta(G) \geq r$ then $\chi_r(G \square H) = max\{\chi(G), \chi(H)\}$.*

The Results

Now, we are ready to show our results on r-dynamic coloring for some special graph operations. Apart from showing the r-dynamic chromatic number we also show the colors $c(v \in V(G))$ for clarity. Some graph operations which have been found in this paper are $P_n + C_m, C_n \square S_m, C_n \otimes S_m, C_n[S_m], C_n \odot S_m, shack(P_n \square C_m, v, s)$, $amal(P_n \square C_m, v, s)$.

Theorem 1.5. *Let G be a joint P_n and C_m. For $n \geq 2$ dan $m \geq 3$, the r-dynamic chromatic number of G is*

$$\chi(P_n + C_m) = \chi_d(P_n + C_m) = \chi_3(P_n + C_m) = \begin{cases} 4, & \text{for } \boldsymbol{m} \text{ even} \\ 5, & \text{for } \boldsymbol{m} \text{ odd} \end{cases}$$

$$\chi_4(P_n + C_m) = \begin{cases} 5, & \text{for } m \equiv 3 (\text{mod } 3) \\ 6, & \text{otherwise} \end{cases}$$

Proof. The graph $P_n + C_m$ is a connected graph with vertex set $V(P_n + C_m)$= $\{x_i; 1 \leq i \leq n\} \cup \{y_j; 1 \leq j \leq m\}$ and $E(P_n + C_m)$= $\{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{y_j y_{j+1}, y_m y_1; 1 \leq j \leq m-1\} \cup \{x_i y_j; 1 \leq i \leq n;\ 1 \leq j \leq m\}$. Thus $p = |V(P_n + C_m)| = n + m, q = |E(G)| = nm + n + m - 1$ and $\Delta(P_n + C_m) = m + 2$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(P_n + C_m) \geq min\{\Delta(P_n + C_m), r\} + 1 = \{m + 2, r\} + 1$.

For $\chi(P_n + C_m) = \chi_d(P_n + C_m) = \chi_3(P_n + C_m)$, define the vertex colouring $c : V(P_n + C_m) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 2$ and $m \geq 3$ as follows:

$$c(x_i) = \begin{cases} 1, \ 1 \leq i \leq n, \ i \text{ odd} \\ 2, \ 1 \leq i \leq n, \ i \text{ even} \end{cases} \qquad c(y_j) = \begin{cases} 3, \ 1 \leq j \leq m, \ j \text{ odd}, \ m \text{ even} \\ 4, \ 1 \leq j \leq m, \ j \text{ even}, \ m \text{ even} \end{cases}$$

$$c(y_j) = \begin{cases} 3, \ 1 \leq j \leq m-1, \ j \text{ odd}, \ m \text{ odd} \\ 4, \ 1 \leq j \leq m-2, \ j \text{ even}, \ m \text{ odd} \\ 5, \ j = m, \ m \text{ odd} \end{cases}$$

It is easy to see that $c : V(P_n + C_m) \rightarrow \{1, 2, \ldots, 4\}$ and $c : V(P_n + C_m) \rightarrow \{1, 2, \ldots, 5\}$, for m even and odd respectively, are proper coloring. Thus, $\chi(P_n + C_m) = 4$ and $\chi(P_n + C_m) = 5$, for m even and odd respectively. By definition, since $min\{|c(N(v))|, \text{ for every } v \in V(P_n + C_m)\} = 3 \leq \delta(P_n + C_m) = 4$, it implies $\chi(P_n + C_m) = \chi_d(P_n + C_m) = \chi_3(P_n + C_m)$.

For $\chi_4(P_n + C_m)$, define the vertex colouring $c : V(P_n + C_m) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 2$ and $m \geq 3$ as follows:
For $m \equiv 3$ (mod 3)

$$c(x_i) = \begin{cases} 1, \ 1 \leq i \leq n, \ i \text{ odd} \\ 2, \ 1 \leq i \leq n, \ i \text{ even} \end{cases}$$

$$c(y_j) = \begin{cases} 3, \ 1 \leq j \leq m, \ j \equiv 5 \ (\text{mod } 3) \\ 4, \ 1 \leq j \leq m, \ j \equiv 4 \ (\text{mod } 3) \\ 5, \ 1 \leq j \leq m, \ j \equiv 3 \ (\text{mod } 3) \end{cases}$$

For $m \equiv 4 \pmod 3$

$$c(x_i) = \begin{cases} 1, \ 1 \le i \le n, \ i \text{ odd} \\ 2, \ 1 \le i \le n, \ i \text{ even} \end{cases}$$

$$c(y_j) = \begin{cases} 3, \ 1 \le j \le m-1, \ j \equiv 5 \pmod 3 \\ 4, \ 1 \le j \le m-1, \ j \equiv 4 \pmod 3 \\ 5, \ 1 \le j \le m-1, \ j \equiv 3 \pmod 3 \\ 6, \ j = m \end{cases}$$

For $m \equiv 5 \pmod 3$

$$c(x_i) = \begin{cases} 1, \ 1 \le i \le n, \ i \equiv 5 \pmod 3 \\ 2, \ 1 \le i \le n, \ i \equiv 4 \pmod 3 \\ 3, \ 1 \le i \le n, \ i \equiv 3 \pmod 3 \end{cases}$$

$$c(y_j) = \begin{cases} 4, \ 1 \le j \le m, \ j \text{ odd}, \ m \text{ even} \\ 5, \ 1 \le j \le m, \ j \text{ even}, \ m \text{ even} \end{cases}$$

$$c(y_j) = \begin{cases} 4, \ 1 \le j \le m-1, \ j \text{ odd}, \ m \text{ odd} \\ 5, \ 1 \le j \le m-2, \ j \text{ even}, \ m \text{ odd} \\ 6, \ j = m, \ m \text{ odd} \end{cases}$$

It is easy to see, for $m \equiv 3 \pmod 3$ $c : V(P_n + C_m) \rightarrow \{1, 2, \dots, 5\}$, and otherwise $c : V(P_n+C_m) \rightarrow \{1, 2, \dots, 6\}$ are proper coloring. Thus, for $m \equiv 3 \pmod 3$, $\chi_4(P_n+C_m) = 5$ and $\chi(P_n + C_m) = 6$ otherwise. By definition, since $min\{|c(N(v))|, \text{ for every } v \in V(P_n + C_m)\} = 4 \le \delta(P_n + C_m) = 4$, it is proved that $\chi_4(P_n + C_m) = 5$. $\square$

Problem 1. *Let G be a joint P_n and C_m. For $n \ge 2$ and $m \ge 3$, determine the r-dynamic chromatic number of G when $r \ge 5$.*

Theorem 1.6. *Let G be a joint W_n and P_m. For $n \ge 3$ dan $m \ge 2$, the r-dynamic chromatic number of G is*

$$\chi(G) = \chi_d(G) = \chi_3(G) = \chi_4(G) \begin{cases} 5, & \textit{for n even} \\ 6, & \textit{for n odd} \end{cases}$$

Proof. The graph $W_n + P_m$ is a connected graph with vertex set $V(W_n + P_m)$= $\{A, x_i, y_j; 1 \le i \le n; 1 \le j \le m\}$ and $E(P_n + C_m)$= $\{Ax_i; 1 \le i \le n\}$ $\cup\{x_ix_{i+1}; 1 \le i \le n-1\}$ $\cup\{x_1x_n\}$ $\cup\{Ay_j; 1 \le j \le m\}$ $\cup\{x_iy_j; 1 \le i \le n-1; 1 \le j \le m\}$ $\cup\{x_ny_j; 1 \le j \le m\}$ $\cup\{y_jy_{j+1}; 1 \le j \le m-1\}$. Thus $p = |V(W_n + P_m)| = n + m + 1, q = |E(G)| = nm + 2n + 2m - 1$ and $\Delta(W_n + P_m) = m + n$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(W_n + P_m) \ge min\{\Delta(W_n + P_m), r\} + 1 = \{m + n, r\} + 1$. Define the vertex coloring $c : V(W_n + P_m) \rightarrow \{1, 2, \dots, k\}$ for $n \ge 3$ and $m \ge 2$ as follows:
For n even

$$c(x_i) = \begin{cases} 1, \ 1 \le i \le n, \ i \text{ even} \\ 2, \ 1 \le i \le n, \ i \text{ odd} \end{cases} \qquad c(y_j) = \begin{cases} 4, \ 1 \le j \le m, \ j \text{ odd} \\ 5, \ 1 \le j \le m, \ j \text{ even} \end{cases}$$

For n odd

$$c(x_i) = \begin{cases} 1, \ 1 \le i \le n-1, \ i \text{ odd} \\ 2, \ 1 \le i \le n-1, \ i \text{ even} \\ 4, \ i = n \end{cases} \qquad c(y_j) = \begin{cases} 5, \ 1 \le j \le m, \ j \text{ odd} \\ 6, \ 1 \le j \le m, \ j \text{ even} \end{cases}$$

It is easy to see that $c : V(W_n+P_m) \to \{1, 2, \ldots, 4\}$ and $c : V(W_n+P_m) \to \{1, 2, \ldots, 5\}$, for n even and odd respectively, is proper coloring. Thus, $\chi(W_n+P_m) = 5$ and $\chi(W_n+P_m) = 6$, for m even and odd respectively. By definition, since $min\{|c(N(v))|, \text{ for every } v \in V(W_n+P_m)\} = 4$, it implies $\chi(W_n+P_m) = \chi_d(W_n+P_m) = \chi_3(W_n+P_m) = \chi_4(W_n+P_m)$. It completes the proof. □

Problem 2. *Let G be a joint W_n and P_m. For $n \ge 2$ and $m \ge 3$, determine the r-dynamic chromatic number of G when $r \ge 5$.*

Theorem 1.7. *Let G be a composition of graph C_n on S_m. For $n \ge 3$ dan $m \ge 3$, the r-dynamic chromatic number of G is*

$$\chi(C_n[S_m]) = \chi_d(C_n[S_m]) = \chi_3(C_n[S_m]) = \begin{cases} 4, \textit{for n even} \\ 6, \textit{for n odd} \end{cases}$$

Proof. The graph $C_n[S_m]$ is a connected graph with vertex set $V(C_n[S_m])$= $\{A_i; 1 \le i \le n\}$ $\cup\{x_{i,j}; 1 \le i \le n \ 1 \le j \le m\}$ and $E(C_n[S_m])$= $\{A_iA_{i+1}; \ 1 \le i \le n-1\}$ $\cup\{A_nA_1\}$ $\cup\{x_{i,j}x_{i+1,j}; \ 1 \le i \le n-1; 1 \le j \le m\}$ $\cup\{x_{n,j}x_{1,j}; \ 1 \le j \le m\}$ $\cup\{A_ix_{i,j}; 1 \le i \le n; 1 \le j \le m\}$ $\cup\{A_ix_{i+1,j}; 1 \le i \le n-1; 1 \le j \le m\}$ $\cup\{A_ix_{i-1,j}; 2 \le i \le n; 1 \le j \le m\}$ $\cup\{A_1x_{n,j}; 1 \le j \le m\}$ $\cup\{A_nx_{1,j}; 1 \le j \le m\}$. Thus $|V(C_n[S_m])| = nm+n$ and $|E(C_n[S_m])| = 4nm+n$ and $\Delta(C_n[S_m]) = 3m+2$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(C_n[S_m]) \ge min\{\Delta(C_n[S_m]), r\}+1 = \{3m+2, r\}+1$. Define the vertex colouring $c : V(C_n[S_m]) \to \{1, 2, \ldots, k\}$ for $n \ge 3$ and $m \ge 3$ as follows:

$$c(A_i) = \begin{cases} 1, \ 1 \le i \le n, \ i \text{ odd} \\ 2, \ 1 \le i \le n, \ i \text{ even} \end{cases}$$

$$c(x_{i,j}) = \begin{cases} 3, \ 1 \le i \le n, \ i \text{ odd}; \ 1 \le j \le m \text{ and } n \text{ even} \\ 4, \ 1 \le i \le n, \ i \text{ odd}; \ 1 \le j \le m \text{ and } n \text{ even} \end{cases}$$

$$c(x_{i,j}) = \begin{cases} 3, \ 1 \le i \le n-2, \ i \text{ odd}; \ 1 \le j \le m \text{ and } n \text{ odd} \\ 4, \ 1 \le i \le n-1, \ i \text{ odd}; \ 1 \le j \le m \text{ and } n \text{ odd} \\ 5, \ i = n \end{cases}$$

It is easy to see that $c : V(C_n[S_m]) \to \{1, 2, \ldots, 4\}$ and $c : V(C_n[S_m]) \to \{1, 2, \ldots, 5\}$, for n even and odd respectively, is proper coloring. Thus, $\chi(C_n[S_m]) = 4$ and $\chi(C_n[S_m]) = 5$, for n even and odd respectively. By definition, since $min\{|c(N(v))|,$
$\text{for every } v \in V(C_n[S_m])\} = 3 \le \delta(C_n[S_m]) = 5$, it implies $\chi(C_n[S_m]) = \chi_d(C_n[S_m]) = \chi_3(C_n[S_m])$. It completes the proof. □

Problem 3. *Let G be a cartesian product of C_n and S_m. For $n \ge 3$ and $m \ge 3$, determine the r-dynamic chromatic number of G when $r \ge 4$.*

Theorem 1.8. *Let G be a crown product of W_n on P_m. For $n \geq 3$ dan $m \geq 2$, the r-dynamic chromatic number of G is*

$$\chi(W_n \odot P_m) = \chi_d(W_n \odot P_m) = \begin{cases} 3, & \text{for } n \text{ even} \\ 4, & \text{for } n \text{ odd} \end{cases}$$

Proof. The graph $W_n \odot P_m$ is a connected graph with vertex set $V(W_n \odot P_m)$= $\{A, x_i, x_{i,j}, y_j; 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(W_n \odot P_m)$= $\{Ax_i; 1 \leq i \leq n\} \cup \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{Ay_j; 1 \leq j \leq m\} \cup \{y_j y_{j+1}; 1 \leq j \leq m-1\} \cup \{x_1 x_n\} \cup \{x_i x_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\} \cup \{x_{i,j} x_{i,j+1}; 1 \leq i \leq n; 1 \leq j \leq m-1\}$. Thus $|V(W_n[P_m])| = nm + n + m + 1$ and $|E(W_n \odot P_m)| = 2nm + n + 2m - 1$ and $\Delta(W_n \odot P_m) = n + m$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(W_n \odot P_m) \geq min\{\Delta(W_n \odot P_m), r\} + 1 = \{n + m, r\} + 1$. Define the vertex coloring $c : V(W_n \odot P_m) \rightarrow \{1, 2, \dots, k\}$ for $n \geq 3$ and $m \geq 2$ as follows: $A = 4$ and

$$c(y_j) = \begin{cases} 1, & 1 \leq j \leq m, \ j \text{ even} \\ 3, & 1 \leq j \leq m, \ j \text{ odd} \end{cases}$$

For n even

$$c(x_{i,j}) = \begin{cases} 1, & 1 \leq i \leq n, \ i \text{ odd}; \ 1 \leq j \leq m, \ j \text{ even} \\ 2, & 1 \leq i \leq n, \ i \text{ even}; \ 1 \leq j \leq m, \ j \text{ even} \\ 3, & 1 \leq j \leq m, \ j \text{ odd}; \ 1 \leq i \leq n \end{cases}$$

$$c(x_i) = \begin{cases} 1, & 1 \leq i \leq n, \ i \text{ even} \\ 2, & 1 \leq i \leq n, \ i \text{ odd} \end{cases}$$

For n odd

$$c(x_{i,j}) = \begin{cases} 1, & 1 \leq i \leq n, \ i \text{ odd}; \ 1 \leq j \leq m, \ j \text{ even} \\ 2, & 1 \leq i \leq n, \ i \text{ even}, 1 \leq j \leq m, \ i \text{ even} \\ 3, & 1 \leq j \leq m-1, \ j \text{ even}; \ 1 \leq i \leq n-1 \\ 4, & 1 \leq j \leq m, \ j \text{ odd}; \ i = n \end{cases}$$

$$c(x_i) = \begin{cases} 1, & 1 \leq i \leq n-1, \ i \text{ even} \\ 2, & 1 \leq i \leq n-1, \ i \text{ odd} \\ 3, & i = n \end{cases}$$

It is easy to see that $c : V(W_n \odot P_m) \rightarrow \{1, 2, \dots, 3\}$ and $c : V(W_n \odot P_m) \rightarrow \{1, 2, \dots, 4\}$, for n even and odd respectively, is proper coloring. Thus, $\chi(W_n \odot P_m) = 3$ and $\chi(W_n \odot P_m) = 4$, for n even and odd respectively. By definition, since $min\{|c(N(v))|, \text{ for every } v \in V(W_n \odot P_m)\} = 2$, it implies $\chi(W_n \odot P_m) = \chi_d(W_n \odot P_m)$. It completes the proof. □

Problem 4. *Let G be a crown product of W_n on P_m. For $n \geq 3$ dan $m \geq 2$, determine the r-dynamic chromatic number of G when $r \geq 3$.*

Theorem 1.9. *Let G be a crown product of C_n on S_m. For $n \geq 3$ dan $m \geq 3$, the r-dynamic chromatic number of G is*

$$\chi(C_n \odot S_m) = \chi_d(C_n \odot S_m) = \begin{cases} 3, & \text{for } n \text{ even} \\ 4, & \text{for } n \text{ odd} \end{cases}$$

Proof. The graph $C_n \odot S_m$ is a connected graph with vertex set $V(C_n \odot S_m)$= $\{A\}$ $\cup\{x_j; 1 \leq j \leq m\}$ $\cup\{y_i; 1 \leq i \leq n\}$ $\cup\{y_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(C_n \odot S_m)$= $\{Ax_j; 1 \leq j \leq m\}$ $\cup\{Ay_i; 1 \leq i \leq n\}$ $\cup\{x_j y_{i,j}; 1 \leq i \leq n; 1 \leq j \leq m\}$ $\cup\{y_i y_{i+1}; 1 \leq i \leq n-1\}$ $\cup\{y_n y_1\}$ $\cup\{y_{i,j} y_{i+1,j}; 1 \leq i \leq n-1; 1 \leq j \leq m\}$ $\cup\{y_{n,j} y_{1,j}; 1 \leq j \leq m\}$. Thus $|V(C_n[S_m])| = nm+n+m+1$ and $|E(C_n \odot S_m)| = 2nm+m+2n$ and $\Delta(C_n \odot S_m) = m+n$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(C_n \odot S_m) \geq min\{\Delta(C_n \odot S_m), r\}+1 = \{m+n, r\}+1$. Define the vertex colouring $c : V(C_n \odot S_m) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 3$ and $m \geq 3$ as follows: $A = 1, c(x_j) = 2,\ 1 \leq j \leq m$ and

For n even

$$c(y_i) = \begin{cases} 2,\ 1 \leq i \leq n,\ i \text{ odd} \\ 3,\ 1 \leq i \leq n,\ i \text{ even} \end{cases};\ c(y_{i,j}) = \begin{cases} 1,\ 1 \leq i \leq n,\ i \text{ odd}, 1 \leq j \leq m \\ 3,\ 1 \leq i \leq n,\ i \text{ even}, 1 \leq j \leq m \end{cases}$$

For n odd

$$c(y_i) = \begin{cases} 2,\ 1 \leq i \leq n-2,\ i \text{ odd} \\ 3,\ 1 \leq i \leq n-1,\ i \text{ even} \\ 4,\ i = n \end{cases};\ c(y_{i,j}) = \begin{cases} 1,\ 1 \leq i \leq n,\ i \text{ odd}, 1 \leq j \leq m \\ 3,\ 1 \leq i \leq n,\ i \text{ even}, 1 \leq j \leq m \\ 4,\ i = n \end{cases}$$

It is easy to see that $c : V(C_n \odot S_m) \rightarrow \{1, 2, \ldots, 3\}$ and $c : V(C_n \odot S_m) \rightarrow \{1, 2, \ldots, 4\}$, for n even and odd respectively, is proper coloring. Thus, $\chi(C_n \odot S_m) = 3$ and $\chi(C_n \odot S_m) = 4$, for n even and odd respectively. By definition, since $min\{|c(N(v))|, \text{ for every } v \in V(C_n \odot S_m)\} = 2 \leq \delta(C_n \odot S_m) = 3$, it implies $\chi(C_n \odot S_m) = \chi_d(C_n \odot S_m)$. It completes the proof. $\square$

Problem 5. *Let G be a crown product of C_n on S_m. For $n \geq 3$ dan $m \geq 3$, determine the r-dynamic chromatic number of G when $r \geq 3$.*

Theorem 1.10. *Let G be a shackle of cartesian product P_n and C_m. For $n \geq 2$ and $m \geq 3$, the r-dynamic chromatic number of G is*

$$\chi(shack(P_n \Box C_m, v, s)) = \chi_d(shack(P_n \Box C_m, v, s)) = \begin{cases} 3,\ \text{for } \boldsymbol{n} \text{ even} \\ 4,\ \text{for } \boldsymbol{n} \text{ odd} \end{cases}$$

Proof. The shackle of cartesian product P_n and C_m, denoted by $shack(P_n \Box C_m, v, s)$, is a connected graph with vertex set V= $\{x^k_{i,j}; 1 \leq i \leq n;\ 1 \leq j \leq m;\ 1 \leq k \leq s\}$ $\cup\{x^k_{n,j}; 1 \leq j \leq m;\ 1 \leq k \leq s\}$ $\cup\{x^r_{n,j}; 1 \leq j \leq m\}$ dan E= $\{x^k_{i,j} x^k_{i,j+1}; 1 \leq i \leq n;\ 1 \leq j \leq m-1;\ 1 \leq k \leq s\}$ $\cup\{x^k_{i,m} x^k_{i,1}; 1 \leq i \leq n;\ 1 \leq k \leq s\}$ $\cup\{x^k_{i,j} x^k_{i+1,j}; 1 \leq i \leq n;\ 1 \leq j \leq m;\ 1 \leq k \leq s\}$. Thus $|V(shack(P_n \Box C_m, v, s))| = nms - s + 1$ and $|E(shack(P_n \Box C_m, v, s))| = 2nms - ns$ and $\Delta(shack(P_n \Box C_m, v, s)) = 6$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(shack(P_n \Box C_m, v, s)) \geq min\{\Delta(shack(P_n \Box C_m, v, s)), r\} + 1 = \{6, r\} + 1$. Define the vertex colouring $c : V(shack(P_n \Box C_m, v, s)) \rightarrow \{1, 2, \ldots, k\}$ for $n \geq 3$ and $m \geq 3$ as follows:

For m even

$$c(x^k_{i,j}) = \begin{cases} 1,\ 1 \leq j \leq m-1,\ j \text{ odd},\ k \text{ odd and } i \text{ odd} \\ 2,\ 1 \leq j \leq m,\ j \text{ even},\ k \text{ odd and } i \text{ odd} \end{cases}$$

$$c(x^k_{i,j}) = \begin{cases} 1,\ 1 \leq j \leq m,\ j \text{ even},\ k \text{ odd and } i \text{ even} \\ 2,\ 1 \leq j \leq m-1,\ j \text{ odd},\ k \text{ odd and } i \text{ even} \end{cases}$$

$$c(x_{i,j}^k) = \begin{cases} 1, \ 1 \le j \le m-1, \ j \text{ odd}, \ k \text{ odd and } i = n \\ 2, \ 1 \le j \le m-1, \ j \text{ even}, \ k \text{ odd and } i = n \end{cases}$$

$$c(x_{i,j}^k) = \begin{cases} 1, \ 1 \le j \le m-1, \ j \text{ even}, \ k \text{ even and } i \text{ odd} \\ 2, \ 1 \le j \le m, \ j \text{ odd}, \ k \text{ even and } i \text{ odd} \end{cases}$$

$$c(x_{i,j}^k) = \begin{cases} 1, \ 1 \le j \le m-1, \ j \text{ odd}, \ k \text{ even and } i \text{ even} \\ 2, \ 1 \le j \le m, \ j \text{ even}, \ k \text{ even and } i \text{ even} \end{cases}$$

$$c(x_{i,j}^k) = \begin{cases} 1, \ 1 \le j \le m-1, \ j \text{ even}, \ k \text{ even and } i = n \\ 2, \ 1 \le j \le m-1, \ j \text{ odd}, \ k \text{ even and } i = n \end{cases}$$

For m odd

$$c(x_{i,j}^k) = \begin{cases} 1, \ 1 \le j \le m-2, \ j \text{ even}, \ k \text{ even and } i \text{ odd} \\ 2, \ 1 \le j \le m-1, \ j \text{ odd}, \ k \text{ even and } i \text{ odd} \\ 3, \ j = m \end{cases}$$

$$c(x_{i,j}^k) = \begin{cases} 1, \ 1 \le j \le m-2, \ j \text{ odd}, \ k \text{ odd and } i \text{ even} \\ 2, \ 1 \le j \le m-1, \ j \text{ even}, \ k \text{ odd and } i \text{ even} \\ 3, \ j = m \end{cases}$$

$$c(x_{i,j}^k) = \begin{cases} 1, \ 1 \le j \le m-2, \ j \text{ even}, \ k \text{ odd and } i = n \\ 2, \ 1 \le j \le m-1, \ j \text{ odd}, \ k \text{ odd and } i = n \\ 3, \ j = m \end{cases}$$

It is easy to see that $c : V(shack(P_n \Box C_m, v, s)) \to \{1, 2\}$ and $c : V(C_n \odot S_m) \to \{1, 2, 3\}$, for m even and odd respectively, are proper coloring. Thus, $\chi(shack(P_n \Box C_m, v, s)) = 2$ and $\chi(shack(P_n \Box C_m, v, s)) = 3$, for m even and odd respectively. By definition, since $min\{|c(N(v))|, \text{ for every } v \in V(shack(P_n \Box C_m, v, s)\}$
$= 1 \le \delta(shack(P_n \Box C_m, v, s)) = 3$, thus we only have $\chi(shack(P_n \Box C_m)) = 2$ and $\chi(shack(P_n \Box C_m, v, s)) = 3$, for m even and odd respectively. It completes the proof. $\square$

Problem 6. *Let G be a shackle of cartesian product P_n and C_m. For $n \ge 2$ and $m \ge 3$, determine the r-dynamic chromatic number of G when $r \ge 2$.*

Theorem 1.11. *Let G be a shackle of joint S_n and P_m. For $n \ge 3$ and $m \ge 2$, the r-dynamic chromatic number of G is*

$$\chi(shack(S_n + P_m, v, s)) = \chi_d(shack(S_n + P_m, v, s)) = \chi_3(shack(S_n + P_m, v, s)) = 4$$

Proof. The shackle of joint S_n and P_m, denoted by $shack(S_n + P_m, v, s)$, is a connected graph with vertex set V= $\{A_k, x_1^k, x_i^k, y_j^k, p; 1 \le i \le n; 1 \le j \le m; 1 \le k \le s\}$ and E= $\{A_k x_i^k; 1 \le i \le n-1; 1 \le k \le s\}$ $\cup\{A_k x_i^{k+1}; 1 \le k \le s\}$ $\cup\{A_s p\}$ $\cup\{y_j^k y_{j+1}^k; 1 \le j \le m-1; 1 \le k \le s\}$ $\cup\{A^k y_j^k; 1 \le j \le m; 1 \le k \le s\}$ $\cup\{x_i^k y_j^k; 1 \le i \le n-1; 1 \le j \le m; 1 \le k \le s\}$ $\cup\{x_1^{k+1} y_j^k; 1 \le j \le m; 1 \le k \le s-1\}$ $\cup\{p y_j^s; 1 \le j \le m\}$. Thus $|V(shack(S_n + P_m, v, s))| = nr + mr + 1$ and $|E(shack(S_n + P_m, v, s))| = 2nms + ns + 2ms - s$ and $\Delta(shack(S_n + P_m, v, s)) = 6$. By Observation 1, the lower bound of r-dynamic chromatic number $\chi_r(shack(S_n + P_m, v, s)) \ge$

$min\{\Delta(shack(S_n+P_m,v,s)),r\}+1=\{6,r\}+1$. Define the vertex coloring $c:V(shack(S_n+P_m,v,s))\rightarrow\{1,2,\dots,k\}$ for $n\geq 3$ and $m\geq 2$ as follows: $c(A^k)=4$

$$c(x_i^k)=\{\ 3,\ 1\leq i\leq n-1;\ 1\leq k\leq s$$

$$c(y_i^k)=\begin{cases}1,\ 1\leq j\leq m,\ j\text{ odd};\ 1\leq k\leq s\\2,\ 1\leq j\leq m,\ j\text{ even};\ 1\leq k\leq s\end{cases}$$

It is easy to see that $c:V(shack(S_n+P_m,v,s))\rightarrow\{1,2,\dots,4\}$ is proper coloring. Thus, $\chi(shack(S_n+P_m,v,s))$
$=4$. By definition, since $min\{|c(N(v))|,\text{ for every }v\in V(shack(S_n+P_m,v,s)\}=3$, it implies $\chi(shack(S_n+P_m))=\chi_d(shack(S_n+P_m))=\chi_3(shack(S_n+P_m))$. It completes the proof. □

Problem 7. *Let G be a shackle of joint S_n and P_m. For $n\geq 3$ and $m\geq 2$, determine the r-dynamic chromatic number of G when $r\geq 4$.*

Conclusions

We have studied the r-dynamic coloring of some graph operations. The results show for each graph operation, its r-dynamic chromatic number has not been obtained completely for all values of r, therefore we left them as open problems for the further study.

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