Rainbow connection number of $C_m \ast P_n$ and $C_m \ast C_n$

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Abstract

Let $G = (V(G), E(G))$ be a nontrivial connected graph. A rainbow path is a path where each edge has different color. A rainbow coloring is a coloring which any two vertices can be joined by at least one rainbow path. For two different vertices, $u, v$ in $G$, geodesic path of $u - v$ is the shortest path of $u - v$. A strong rainbow coloring is a coloring which any two vertices can be joined by at least one rainbow geodesic. A rainbow connection number of a graph, denoted by $rc(G)$, is the smallest number of color required for graph $G$ to be rainbow connected. The strong rainbow color number, denoted by $src(G)$, is the least number of color which is needed to color every geodesic path in the graph $G$ to be rainbow. In this paper, we will determine the rainbow connection and strong rainbow connection numbers for Corona Graph $C_m \ast P_n$ and $C_m \ast C_n$.

Keywords: rainbow connection number, strong rainbow connection number, corona product

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1. Introduction

The concept of rainbow connection of a graph was first introduced by Chartrand, Johns, McKeean and Zhang [2] in 2006. Let $G = (V(G), E(G))$ be a nontrivial connected graph. Define a coloring $c : E(G) \rightarrow \{1, 2, \ldots, k\}, k \in N$, where two neighbor edges may have the same color. A path $u - v$ path $P$ in $G$ is called a rainbow path if there are no two edges in $P$ of the same color. A graph $G$ is called rainbow connected if every two different vertices in $G$ are connected by the rainbow path.

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The edge coloring that causes $G$ is rainbow connected is said to be rainbow coloring. Obviously, if $G$ is rainbow connected, then $G$ is connected. Each connected graph has a trivial edge coloring so that $G$ is rainbow connected, where each edge has different colors. The rainbow connection number of the connected graph $G$, denoted by $rc(G)$, is the smallest number of colors required to make the graph $G$ to be rainbow connected [3]. A rainbow coloring that uses $rc(G)$ colors is called minimum rainbow coloring.

Let $c$ be a rainbow coloring of a connected graph $G$. For two vertices $u$ and $v$ of $G$, a rainbow $u - v$ geodesic path in $G$ is a rainbow $u - v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between $u$ and $v$ (the length of a shortest $u - v$ path in $G$). The graph $G$ is called strongly rainbow-connected if $G$ has a rainbow $u - v$ geodesic path for every pair of vertices $u$ and $v$ of $G$. In this case, the coloring $c$ is called a strong rainbow coloring of $G$. The minimum $k$ for which there exists a coloring $c : E(G) \rightarrow \{1, 2, \ldots, k\} | k \in \mathbb{N} \}$ of the edges of $G$ such that $G$ is strongly rainbow-connected is called the strong rainbow connection number $src(G)$ of $G$ [3]. A strong rainbow coloring of $G$ using $src(G)$ colors is called a minimum strong rainbow coloring of $G$. Thus $rc(G) \leq src(G)$ for every connected graph $G$ [2]. Furthermore, if $G$ is a connected nontrivial graph with size $m$ and $diam(G) = \max\{d(u, v) | u, v \in V(G)\}$, then

$$diam(G) \leq rc(G) \leq src(G) \leq m$$

In this research, we examine the rainbow connection and strong rainbow connection for Corona Graph $C_m \odot P_n$ and $C_m \odot C_n$.

2. Known Results

**Definition 2.1.** (Chartrand and Lesniak [1]) Path graph $P_n$ is a connected graph consisting of $n$ vertices where degree of two vertices are one and degree of $n - 2$ vertices are two.

**Definition 2.2.** (Chartrand and Lesniak [1]) The cycle graph $C_m$ is a connected graph that forms a circle with degree of $m$-vertices are equal to two.

**Definition 2.3.** (Kaladevi and Kavitha [4]) Let $G$ be a graph with $n$ vertices, $v_1, v_2, \ldots, v_n$, and $H$ is a graph with $m$ vertices. The corona operation of two graphs $G$ and $H$, $G \odot H$, is defined as the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of graph $H$ and then joining the $i^{th}$ vertex of $G$ to every vertex in the $i^{th}$ copy of $H$.

**Lemma 2.1.** (Chartrand et al. [2]) $rc(G) = src(G) = 1$ if and only if $G$ is a complete graph.

**Proposition 2.1.** (Chartrand et al. [2]) Let $C_n$ be a cycle graph. For each integer $n \geq 4$, $rc(C_n) = src(C_n) = \lceil \frac{n}{2} \rceil$.

A fan graph $F_n$ is a corona graph $K_1 \odot P_n$ where each vertex in $P_n$ is connected to the vertex in $K_1$. A wheel graph can be considered as a corona product of $K_1$ with a cycle graph $C_n$. Thus a wheel graph is $K_1 \odot C_n$.

**Proposition 2.2.** (Syafrizal et al. [5]) For $n \geq 2$, the rainbow connection number

$$rc(F_n) = \begin{cases} 1, & n = 2; \\ 2, & 3 \leq n \leq 6; \\ 3, & n \geq 7. \end{cases}$$
Proposition 2.3. (Syafirizal et al. [5]) For integers \( n \geq 2 \), the strong rainbow connection number of the fan graph \( F_n \)

\[
src(F_n) = \begin{cases}
1, & \text{for } n = 2 \\
2, & \text{for } 3 \leq n \leq 6 \\
\lceil \frac{n}{3} \rceil, & \text{for } n \geq 7
\end{cases}
\]

Proposition 2.4. (Chartrand et al. [2]) For \( n \geq 3 \), the rainbow connection number of the wheel graph \( W_n \) is

\[
rc(W_n) = \begin{cases}
1, & n = 3; \\
2, & 4 \leq n \leq 6; \\
3, & n \geq 7.
\end{cases}
\]

Proposition 2.5. (Chartrand et al. [2]) For integers \( n \geq 3 \), the rainbow connection number of the wheel graph \( W_n \) is

\[
src(W_n) = \left\lceil \frac{n}{3} \right\rceil
\]

3. Main Results

3.1. Corona Graph \( C_m \odot P_n \)

Let \( G = C_m \odot P_n \) then \( V(G) = V(C_m) \cup V(mP_n) \) where \( V(C_m) = \{v_1, v_2, \ldots, v_m\} \) and \( V(mP_n) = \{v_1^1, v_2^1, \ldots, v_m^1\} \cup \{v_1^2, v_2^2, \ldots, v_m^2\} \cup \cdots \cup \{v_1^n, v_2^n, \ldots, v_m^n\} \). 

\[
E(G) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_m, v_1)\} \cup \{(v_1^1, v_2^1), (v_2^1, v_3^1), \ldots, (v_1^{(n-1)}, v_1^n)\} \cup \{(v_1^2, v_2^2), (v_2^2, v_3^2), \ldots, (v_2^{(n-1)}, v_2^n)\} \cup \cdots \cup \{(v_1^n, v_2^n), (v_2^n, v_3^n), \ldots, (v_m^{(n-1)}, v_m^n)\} \cup \{(v_1, v_1^1), (v_1^1, v_1^2), \ldots, (v_1, v_1^n)\} \cup \{(v_2, v_1^1), (v_2, v_1^2), \ldots, (v_2^n, v_1^n)\}
\]

Graph \( C_m \odot P_n \) is a corona graph where \( C_m \) can be considered as its center and every vertex \( v_i \) in \( C_m \) is connected to every vertex in \( P_n \).

Theorem 3.1. The rainbow connection number of corona graph \( C_m \odot P_n \) is

\[
rc(C_m \odot P_n) = \begin{cases}
4, & \text{for } m = 3, n \geq 2 \\
\left\lceil \frac{m}{2} \right\rceil + 3, & \text{for } m > 3, n \geq 2
\end{cases}
\]

Proof. Using the similar coloring of \( C_n \) as in Chartrand et al. [1], and combine it with the coloring for \( F_n \) as in Syafirizal et al. [5], we find the \( rc(C_m \odot P_n) \) in two cases.

Case 1. For \( m = 3, n \geq 2 \), \( rc(C_3 \odot P_n) = 4 \).

Since \( C_3 = K_3 \), then according to Lemma 2.1, \( rc(C_3) = 1 \). Based on Proposition 2.2, we have

\[
rc(F_n) = \begin{cases}
1, & n = 2; \\
2, & 3 \leq n \leq 6; \\
3, & n \geq 7.
\end{cases}
\]

Suppose \( v_a, v_b \in V(C_3 \odot P_n) \) then there are 4 cases to determine \( rc(C_3 \odot P_n) \).
Case 1.1. $v_a, v_b \in V(C_3)$

Based on Lemma 2.1, we obtained that $rc(V(C_3)) = 1$. Then the length of rainbow path $v_a - v_b$ is equal to one.

Case 1.2. $v_a \in V(C_3)$, $v_b \in V(P_n^i); i = 1, 2, 3$

Based on Proposition 2.2, since $v_i$ is equal to one of the center vertex $v_p$, we obtained that the length of rainbow path is equal to $r$, where

$$r = \begin{cases} 
1, & n = 2; \\
2, & 3 \leq n \leq 6; \\
3, & n \geq 7.
\end{cases}$$

Case 1.3. $v_a \in V(C_3)$, $v_b \in V(P_n^i); i = 1, 2, 3$. Since $v_a$ is not equal to $v_p$, so there is a path $v_b - v_p - v_a$, with $v_a \in V(C_3)$, then the length of rainbow path $v_a - v_b$ is equal to two.

Case 1.4. $v_a \in V(P_n^i)$, $v_b \in V(P_n^j); i, j = 1, 2, 3$

Based on Definitions 2.1 and Definitions 2.2, it can be concluded that vertices $\{v_1, v_2, v_3\}$ in $C_3$ is the central of fan graph and its are vertices of cycle $(C_3)$. Thus the vertices in the three fan subgraphs (note that the $rc(F_n) = 3$) require only additional one color to have a rainbow path. Thus the minimum length of the rainbow path $v_a - v_b$ is equal to four.

From the four cases, it is proved that $rc(C_3 \odot P_n) = 4$.

Case 2. For $m > 3, n \geq 2$, $rc(C_m \odot P_n) = \lceil \frac{m}{2} \rceil + 3$. Based on Proposition 2.1, we obtained that $rc(C_m) = \lceil \frac{m}{2} \rceil$. Based on Proposition 2.2, we know that $rc(F_n) = \begin{cases} 
1, & n = 2; \\
2, & 3 \leq n \leq 6; \\
3, & n \geq 7.
\end{cases}$ We will show that $rc(C_m \odot P_n) = \lceil \frac{m}{2} \rceil + 3$, for $n \geq 2$.

\[\text{Figure 1. The coloring illustration of } C_m \odot P_n \text{ Graph}\]

Figure 1 shows the illustration of the coloring of graph $C_m \odot P_n$. From Figure 1, it can be seen that:
The vertices of \( P_n^i, i = 1, 2, \ldots, m \) which is connected to vertex \( v_i \) in the cycle \( C_m \), is formed fan subgraphs of \( C_m \odot P_n \). Let \( v_a, v_b \in V(C_m \odot P_n) \) then there are four cases to determine \( rc(C_m \odot P_n) \).

- **Case 2.1.** \( v_a, v_b \in V(C_m) \)
  
  Based on Proposition 2.1, we obtained that \( rc(C_m) = \lceil \frac{m}{2} \rceil \). Thus the minimum length of rainbow path \( v_a - v_b \) is equal to \( \lceil \frac{m}{2} \rceil \).

- **Case 2.2.** \( v_a \in V(C_m), v_b \in V(P_n^i); i = 1, 2, 3, \ldots, m \)
  
  Based on Proposition 2.2, since \( v_a \) is equal to the center vertex \( v_p \) for one \( p \in \{v_1, \ldots, v_n\} \), and
  
  \[
  rc(F_n) = \begin{cases} 
  1, & n = 2; \\
  2, & 3 \leq n \leq 6; \\
  3, & n \geq 7.
  \end{cases}
  \]

  Thus we know that the maximum length of rainbow path of \( v_a - v_b \) is equal to \( \lceil \frac{m}{2} \rceil \).

- **Case 2.3.** \( v_a \in V(C_m), v_b \in V(P_n^i); i = 1, 2, 3, \ldots, m \)
  
  Since \( v_a \) is not equal to \( v_p \), then there is a path \( v_b - v_p - \ldots - v_a \), with \( v_a \in V(C_m) \), then the length of path \( v_a - v_p - \ldots - v_b \) is equal to \( \lceil \frac{m}{2} \rceil + 1 \).

- **Case 2.4.** \( v_a \in V(P_n^i), v_b \in V(P_n^j); i, j = 1, 2, 3, \ldots, m \).
  
  Based on Definitions 2.1 and Definitions 2.2, it can be concluded that in \( C_m \odot P_n \), the central of fan graphs \( \{v_1, v_2, v_3, \ldots, v_m\} \) formed a cycle \( (C_m) \). Thus the vertices in \( m \)-fan graphs require only three additional colors.

From the four cases, it is proved that \( rc(C_m \odot P_n) = \lceil \frac{m}{2} \rceil + 3 \).

\[\square\]

**Theorem 3.2.** **Strong rainbow connection number of corona graph** \( C_m \odot P_n \).

\[
src(C_m \odot P_n) = \begin{cases} 
\lceil \frac{n}{3} \rceil \cdot 3 + 1, & \text{for } m = 3, n \geq 2 \\
\lceil \frac{n}{3} \rceil \cdot 3 + \lceil \frac{m}{2} \rceil, & \text{for } m > 3, n \geq 2
\end{cases}
\]

**Proof.** Using the similar coloring of \( C_n \) as in Chartrand et al.[1], and combine with the coloring for \( F_n \) as in Syafrizalet al.[5], we prove the rc of \( C_m \odot P_n \) in two parts. The first part is for \( m = 3 \) and the second part is for the case \( m > 3 \) as follows.

**Case 1** For \( m = 3, n \geq 2 \), \( src(C_m \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1 \). Based on Lemma 2.1, we obtained that \( src(C_3) = 1 \). Based on Proposition 2.3, we know that

\[
src(F_n) = \begin{cases} 
1, & \text{for } n = 2 \\
2, & \text{for } 3 \leq n \leq 6 \\
\lceil \frac{n}{3} \rceil, & \text{for } n \geq 7
\end{cases}
\]

We will show that \( src(C_3 \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1 \), for \( m = 3, n \geq 2 \).
Graph $C_3 \odot P_n$ has three fan subgraphs whose respective center vertices are $\{v_1, v_2, v_3\}$, and its are the vertices of the cycle $C_3$. Suppose $v_a, v_b \in V(C_3 \odot P_n)$ then there are four cases to determine $\text{src}(C_3 \odot P_n)$

- Case 1.1. $v_a, v_b \in V(C_3)$
  Based on Lemma 2.1, we obtained that $rc(C_3) = 1$.

- Case 1.2. $v_a \in V(C_3), v_b \in V(P^n_i); i = 1, 2, 3$
  Based on Proposition 2.4, since $v_a$ is equal to center vertex $v_p$, we know that
  $$\text{src}(F_n) = \begin{cases} 1, & \text{for } n = 2 \\ 2, & \text{for } 3 \leq n \leq 6 \\ \lceil \frac{n}{3} \rceil, & \text{for } n \geq 7 \end{cases}$$

- Case 1.3. $v_a \in V(C_3), v_b \in V(P^n_i); i = 1, 2, 3$ Since $v_a$ is not equal to $v_p$, then there is a path $v_b - v_p - v_a$, with $v_a \in V(C_3)$. Thus the length of the geodesic path is equal to two.

- Case 1.4. $v_a \in V(P^n_i), v_b \in V(P^n_j); i, j = 1, 2, 3$
  Consider the two fan subgraphs with center vertices $v_i$ and $v_j$ which are connected together. Since the two graphs are connected, then there is one edge connecting the two centers $v_i$ and $v_j$ of the fan so that its geodesic path is three.
  $\text{src}$ is a rainbow connection that requires the number of colors on the edges is calculating for the geodesic path. This causes each of the fan subgraph to have a different color. Based on Lemma 2.3, by the same coloring that one color can only be used at most three times. Color the spokes of $P^n_i \neq P^n_j \neq P^n_k$, with the additional color to the connecting edge of each $F_n$. Thus, we obtained $\text{src}(C_3 \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$.

It is proved that $\text{src}(C_3 \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$.

**Case 2** For $m > 3, n \geq 2$, $\text{src}(C_m \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$. Based on Proposition 2.1, we obtained that $\text{src}(C_m) = \lceil \frac{m}{2} \rceil$. Based on Proposition 2.4, we know that

$$\text{src}(F_n) = \begin{cases} 1, & \text{for } n = 2 \\ 2, & \text{for } 3 \leq n \leq 6 \\ \lceil \frac{n}{3} \rceil, & \text{for } n \geq 7 \end{cases}$$

We will show that $\text{src}(C_m \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$, for $n \geq 2$.

From the general form (illustrated in Figure 2), we can see that:
$C_m \odot P_n$ has $m$ fan subgraphs which its respective center vertices are $\{v_1, v_2, v_3, \ldots, v_m\}$. Its are vertices of the cycle $C_m$. Suppose that $v_a, v_b \in V(C_m \odot P_n)$ then there are four cases to find $\text{src}(C_m \odot P_n)$.

- Case 2.1. $v_a, v_b \in V(C_m)$
  Based on Proposition 2.1, we knew that $\text{src}(C_m) = \lceil \frac{m}{2} \rceil$. Thus the length of the geodesic path from $v_a - v_b$ is equal to $\lceil \frac{m}{2} \rceil$
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**Figure 2. src** $C_m \odot P_n$ Graph

- **Case 2.2.** $v_a \in V(C_m), v_b \in V(P^i_n); i = 1, 2, 3, \ldots, m$

  Based on Proposition 2.3, since $v_a$ equal to its center vertex $v_p$, we obtained that the length of the geodesic path is equal to

  $$\text{src}(F_n) = \begin{cases} 
  1, & \text{for } n = 2 \\
  2, & \text{for } 3 \leq n \leq 6 \\
  \lceil \frac{n}{3} \rceil, & \text{for } n \geq 7
  \end{cases}$$

- **Case 2.3.** $v_a \in V(C_m), v_b \in V(P^i_n); i = 1, 2, 3, \ldots, m$

  Since $v_a$ is not equal to $v_p$, then there is a path $v_b - v_p - \ldots - v_a$, with $v_a \in V(C_m)$. Thus the length of the geodesic path is equal to $\lceil \frac{m}{2} \rceil + 1$.

- **Case 2.4.** $v_a \in V(P^i_n), v_b \in V(P^j_n); i, j = 1, 2, 3, \ldots, m$

  Consider the two fan subgraphs which their center is adjacent. Then there is only one edge connecting these two center $v_i$ and $v_{i+1}$ of the two fans so that its geodesic path is maximum three.

  Finding the strong rainbow coloring number for $C_m \odot P_n$ needs to guarantee that every pair of vertices where the vertex come from the different fan subgraph, connect with the rainbow geodesic path. This causes each fan subgraph having different color. Based on Lemma 2.3, and using the same coloring as in the proof of Lemma 2.3, we know that one color can only be used at most three times.

  Color the rim of $P^i_n \neq P^j_n \neq P^k_n$, with the additional one color. Then the length of the geodesic path is equal to $(\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$.

  Based on the four cases, it is proved that the $\text{src}(C_m \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$. 

\[\square\]
Theorem 3.3. Rainbow connection cumber of corona graph \( C_m \odot C_n \).

\[
rc(C_m \odot C_n) = \begin{cases} 
4, & \text{for } m = 3, n \geq 3 \\
\left\lceil \frac{m}{2} \right\rceil + 3, & \text{for } m > 3, n \geq 3
\end{cases}
\]

Proof. Using the similar coloring of \( C_n \) as in Chartrand et al.[1], which is illustrated in Figure 3 and Figure 4, we divide the proof in two cases.

Case 1. For \( m = 3, n \geq 3 \), \( rc(C_3 \odot C_n) = 4 \).

Since \( C_3 = K_3 \) then according to Lemma 2.1, \( rc(C_3) = 1 \). Based on Proposition 2.4, we have

\[
rc(W_n) = \begin{cases} 
1, & n = 3; \\
2, & 4 \leq n \leq 6; \\
3, & n \geq 7.
\end{cases}
\]

It will be shown that \( rc(C_3 \odot C_n) = 4 \), for \( n \geq 3 \).

Consider the coloring as illustrated in Figure 3.

From the Figure 3, it can be seen that the graph \( C_3 \odot C_n \) has three wheel subgraphs, where the respective center vertices \( \{v_1, v_2, v_3\} \) are vertices of a cycle \( C_3 \). Suppose that \( v_a, v_b \in V(C_3 \odot C_n) \) then there are four cases to considered for finding the \( rc(C_3 \odot C_n) \)

- Case 1.1. \( v_a, v_b \in V(C_3) \)
  Based on Lemma 2.1, we knew that \( rc(C_3) = 1 \). Thus the length of the rainbow path \( v_a - v_b \) is equal to one.

- Case 1.2. \( v_a \in V(C_3), v_b \in V(C_n^i); i = 1, 2, 3. \)
  Based on Proposition 2.4, since \( v_a \) is equal to \( v_p \) (center vertex of one of a wheel subgraph), we obtained that the length of the rainbow path \( v_a - v_b \) is equal to \( r \), where

\[
r = \begin{cases} 
1, & n = 3; \\
2, & 4 \leq n \leq 6; \\
3, & n \geq 7.
\end{cases}
\]
It is proved that \( rc(C_3 \odot C_n) = 4 \).

**Case 2.** For \( m > 3, n \geq 3 \), \( rc(C_m \odot C_n) = \lceil \frac{m}{2} \rceil + 3 \).

Based on Proposition 2.1, we knew that \( rc(C_m) = \lceil \frac{m}{2} \rceil \). Based on Proposition 2.4, we have

\[
rc(W_n) = \begin{cases} 
1, & n = 3; \\
2, & 4 \leq n \leq 6; \quad \text{We will show that } rc(C_m \odot C_n) = \lceil \frac{m}{2} \rceil + 3, \text{ for } n \geq 3. \\
3, & n \geq 7. 
\end{cases}
\]

Figure 4 gives the illustration of coloring of \( C_m \odot C_n \) graph.

From the Figure 4, it can be seen that \( C_m \odot C_n \) has \( m \) wheel subgraphs, where its respective center vertices \( \{v_1, v_2, \ldots, v_m\} \) are the vertices of the center cycle \( C_m \). Suppose that \( v_a, v_b \in V(C_m \odot C_n) \), then there are four cases to determine \( rc(C_m \odot C_n) \)

- **Case 2.1.** \( v_a, v_b \in V(C_m) \)
  Based on Proposition 2.1, we obtained that \( rc(C_m) = \lceil \frac{m}{2} \rceil \)

- **Case 2.2.** \( v_a \in V(C_m), v_b \in V(P_n^i); i = 1, 2, 3, \ldots, m \)
  Based on Proposition 2.4, since \( v_a \) is equal to \( v_p \) (one of the vertices in the center cycle), we obtained that
  
  the length of the rainbow path is equal to
  \[
  \begin{cases} 
  1, & n = 3; \\
  2, & 4 \leq n \leq 6; \\
  3, & n \geq 7. 
  \end{cases}
  \]

- **Case 2.3.** \( v_a \in V(C_m), v_b \in V(C_n^i); i = 1, 2, 3, \ldots, m \)
  Since \( v_a \) is not equal to \( v_p \), so there is a path \( v_b - v_p - \ldots - v_a \), with \( v_a \in V(C_m) \), then the length of the rainbow path is maximum \( \lceil \frac{m}{2} \rceil + 1 \).

- **Case 2.4.** \( v_a \in V(C_n^i), v_b \in V(C_n^j); i, j = 1, 2, 3, \ldots, m \)
  Based on Definitions 2.1 and Definitions 2.2, it can be concluded that \( C_m \odot P_n \) has the central vertices, \( \{v_1, v_2, \ldots, v_m\} \), of the fan subgraph graph, which they are the vertices of the cycle \( (C_m) \). Thus the \( m \)-fan subgraphs, where each has \( rc = \lceil \frac{m}{2} \rceil \), require only three additional colors.

The four cases proved that \( rc(C_m \odot C_n) = \lceil \frac{m}{2} \rceil + 3 \).
Figure 4. Coloring illustration of $C_m \odot C_n$
Theorem 3.4. Strong rainbow connection number of corona graph $C_m \circ C_n$.

$$src(C_m \circ C_n) = \begin{cases} 
\left(\left\lceil \frac{n}{3} \right\rceil \cdot 3 \right) + 1, & \text{for } m = 3, n \geq 3 \\
\left(\left\lceil \frac{n}{3} \right\rceil \cdot 3 \right) + \left\lceil \frac{m}{2} \right\rceil, & \text{for } m > 3, n \geq 3 
\end{cases}$$

Proof. Similar with the previous theorems, the proof will be divided into two cases.

Case 1. For $m = 3, n \geq 3$, $src(C_m \circ P_n) = \left(\left\lceil \frac{n}{3} \right\rceil \cdot 3 \right) + 1$. Based on Lemma 2.1, we knew that $src(C_3) = 1$. Based on Proposition 2.5, we have

$$src(W_n) = \left\lceil \frac{n}{3} \right\rceil$$

We will show that $src(C_3 \circ C_n) = \left(\left\lceil \frac{n}{3} \right\rceil \cdot 3 \right) + 1$, for $m = 3, n \geq 3$.

![Figure 5. Coloring illustration of $C_3 \circ C_n$](https://example.com/figure5)

From Figure 5, it can be seen that $C_3 \circ C_n$ has three wheel subgraphs where their vertex centers $\{v_1, v_2, v_3\}$ are the vertices of $C_3$. Suppose that $v_a, v_b \in V(C_3 \circ C_n)$, then there are four cases to determine $src(C_3 \circ C_n)$

- Case 1.1. $v_a, v_b \in V(C_3)$
  Based on Lemma 2.1, we knew that $rc(C_3) = 1$. Thus the geodesic path has length one.

- Case 1.2. $v_a \in V(C_3), v_b \in V(C_n^i); i = 1, 2, 3$
  Based on Proposition 2.5, since $v_a$ is equal to one of the vertex center $v_p$, we obtained that the length of geodesic path from $v_a$ to $v_b$ is equal to $\left\lceil \frac{n}{3} \right\rceil$
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- Case 1.3. $v_a \in V(C_3), v_b \in V(C_n^i); i = 1, 2, 3$. Since $v_a$ is not equal to the vertex center $v_p \in \{v_i, v_2, v_3\}$, then there is a path $v_b - v_p - v_a$, with $v_a \in V(C_3)$. Thus the length of the geodesic path is equal to three.

- Case 1.4. $v_a \in V(C_n^i), v_b \in V(C_n^j); i, j = 1, 2, 3$

Look at the two wheel subgraphs, which has center vertex $v_i$ and $v_{i+1}$. These two center vertices are adjacent. Then the two wheel subgraphs are also connected and the length of the geodesic graph is equal to three.

The rainbow geodesic path requires the number of colors are equal to its length. This fact make the color of each wheel has to be different. Based on Lemma 2.3, one color can only be used at most three times in each wheel subgraph.

Thus, we will have the length of geodesic rainbow path is $(\lceil \frac{n}{3} \rceil \cdot 3) + 1$.

From the 4 cases, we prove that $src(C_3 \odot C_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$.

Case 2. For $m > 3, n \geq 3$, $src(C_m \odot C_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$. Based on Proposition 2.1, we obtained $src(C_m) = \lceil \frac{m}{2} \rceil$. Based on Proposition 2.5,

$$src(W_n) = \frac{n}{3}$$

We will show that $src(C_m \odot C_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$, for $n \geq 3$.

Figure 6. Coloring illustration of $C_m \odot C_n$

From the general form which can be seen in Figure 6, that we have $C_m \odot C_n$ has $m$ wheel subgraphs. The wheel subgraphs have its center vertices $\{v_1, v_2, \ldots, v_m\}$ which are vertices of $C_m$. Suppose that $v_a, v_b \in V(C_m \odot C_n)$ then there are four cases to determine $src(C_m \odot C_n)$
• Case 2.1. \(v_a, v_b \in V(C_m)\)
Based on Proposition 2.1, we knew that \(rc(C_m) = \lceil \frac{m}{2} \rceil\). Thus the length of the geodesic path is \(\lceil \frac{m}{2} \rceil\).

• Case 2.2. \(v_a \in V(C_m), v_b \in V(C_n^i); i = 1, 2, 3, \ldots, m\)
Based on Proposition 2.5, since \(v_a\) is equal to \(v_p\) (one of the center vertex), we knew that \(src(W_n) = \lceil \frac{n}{3} \rceil\), which is the same with the length of geodesic path from \(v_a\) to \(v_b\).

• Case 2.3. \(v_a \in V(C_m), v_b \in V(C_n^i); i = 1, 2, 3, \ldots, m\)
Since \(v_a\) is not equal to one of the center vertex \(v_p\), then there is a path \(v_b \rightarrow v_p \rightarrow \ldots \rightarrow v_a\), with \(v_a \in V(C_m)\). Thus the length of the geodesic path is \(\lceil \frac{m}{2} \rceil + 1\).

• Case 2.4. \(v_a \in V(C_n^i), v_b \in V(C_n^j); i, j = 1, 2, 3, \ldots, m\)
Consider the two wheel subgraphs with center vertices are adjacent, which are \(v_i\) and \(v_i + 1\), respectively. Thus the length of the geodesic rainbow path is equal to three. Based on Lemma 2.3, the same color only can be used three times in the spoke of the wheel subgraphs. Color the rim of the wheels with the aditional color to connect the vertices in outher cyccle of the wheel.Thus we obtained the length of the geodesic rainbow path is equal to \((\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil\).

From the four cases, we proved that \(src(C_m \odot C_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil\).

4. Conclusion

From the discussion that we have in this paper, it can be concluded that

1. Rainbow connection number for corona graph \(C_m \odot P_n\) and \(C_m \odot C_n\).
\[
rc(C_m \odot P_n) = \begin{cases} 
4, & \text{for } m = 3, n \geq 2 \\
\lceil \frac{m}{2} \rceil + 3, & \text{for } m > 3, n \geq 2 
\end{cases}
\]
\[
rc(C_m \odot C_n) = \begin{cases} 
4, & \text{for } m = 3, n \geq 3 \\
\lceil \frac{m}{2} \rceil + 3, & \text{for } m > 3, n \geq 3 
\end{cases}
\]

2. Strong rainbow connection number for corona graph \(C_m \odot P_n\) and \(C_m \odot C_n\).
\[
src(C_m \odot P_n) = \begin{cases} 
(\lceil \frac{n}{3} \rceil \cdot 3) + 1, & \text{for } m = 3, n \geq 2 \\
(\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil, & \text{for } m > 3, n \geq 2 
\end{cases}
\]
\[
src(C_m \odot C_n) = \begin{cases} 
(\lceil \frac{n}{3} \rceil \cdot 3) + 1, & \text{for } m = 3, n \geq 3 \\
(\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil, & \text{for } m > 3, n \geq 3 
\end{cases}
\]
3. $C_m \circ C_n$ has the same rc and src as $C_m \circ P_n$, since both of the graphs have the same characteristic where there is a center vertex $v_p$ in wheel graph and fan graph. Thus, they have the same coloring type for rainbow coloring.

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References


