Indonesian Journal of Combinatorics **3** (2) (2019), 95–108

INDONESIAN JOURNAL OF COMBINATORICS

Rainbow connection number of $C_m \odot P_n$ and $C_m \odot C_n$

Alfi Maulani, Soya F.Y.O. Pradini, Dian Setyorini, Kiki A. Sugeng

Department of Mathematics, FMIPA Universitas Indonesia, Kampus UI, Depok 16424, Indonesia

maulanialfi@gmail.com, soya.febeauty@gmail.com, diandeeice@gmail.com, kiki@sci.ui.ac.id

Abstract

Let $G = (V(G), E(G))$ be a nontrivial connected graph. A rainbow path is a path where each edge has different color. A rainbow coloring is a coloring which any two vertices can be joined by at least one rainbow path. For two different vertices, u, v in G, geodesic path of $u - v$ is the shortest path of $u - v$. A strong rainbow coloring is a coloring which any two vertices can be joined by at least one rainbow geodesic. A rainbow connection number of a graph, denoted by $rc(G)$, is the smallest number of color required for graph G to be rainbow connected. The strong rainbow color number, denoted by $src(G)$, is the least number of color which is needed to color every geodesic path in the graph G to be rainbow. In this paper, we will determine the rainbow connection and strong rainbow connection numbers for Corona Graph $C_m \odot P_n$ and $C_m \odot C_n$.

Keywords: rainbow connection number, strong rainbow connection number, corona product
Mathematics Subject Classification : 05C15, 05C40
DOI: 10.19184/ijc.2019.3.2.3

1. Introduction

The concept of rainbow connection of a graph was first introduced by Chartrand, Johns, McKeon and Zhang [2] in 2006. Let $G = (V(G), E(G))$ be a nontrivial connected graph. Define a coloring $c : E(G) \to \{1, 2, \ldots, k\}, k \in N$, where two neighbor edges may have the same color. A path $u - v$ path P in G is called a rainbow path if there are no two edges in P of the same color. A graph G is called rainbow connected if every two different vertices in G are connected by the rainbow path.

Received: 31 Dec 2017, Revised: 21 Mar 2019, Accepted: 30 Dec 2019.

The edge coloring that causes G is rainbow connected is said to be rainbow coloring. Obviously, if G is rainbow connected, then G is connected. Each connected graph has a trivial edge coloring so that G is rainbow connected, where each edge has different colors. The rainbow connection number of the connected graph G, denoted by $rc(G)$, is the smallest number of colors required to make the graph G to be rainbow connected [3]. A rainbow coloring that uses $rc(G)$ colors is called minimum rainbow coloring.

Let c be a rainbow coloring of a connected graph G. For two vertices u and v of G, a rainbow $u-v$ geodesic path in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v (the length of a shortest $u-v$ path in G). The graph G is called strongly rainbow-connected if G has a rainbow $u-v$ geodesic path for every pair of vertices u and v of G. In this case, the coloring c is called a strong rainbow coloring of G. The minimum k for which there exists a coloring $c: E(G) \to \{1,2,\ldots,k | k \in N\}$ of the edges of G such that G is strongly rainbow-connected is called the strong rainbow connection number $src(G)$ of G [3]. A strong rainbow coloring of G using $src(G)$ colors is called a minimum strong rainbow coloring of G. Thus $rc(G) \leq src(G)$ for every connected graph G [2]. Furthermore, if G is a connected nontrivial graph with size m and $diam(G) = max\{d(u,v) | u,v \in V(G)\}$, then

$$diam(G) \leq rc(G) \leq src(G) \leq m$$

In this research, we examine the rainbow connection and strong rainbow connection for Corona Graph $C_m \odot P_n$ and $C_m \odot C_n$.

2. Known Results

Definition 2.1. (Chartrand and Lesniak [1]) Path graph P_n is a connected graph consisting of n vertices where degree of two vertices are one and degree of $n-2$ vertices are two.

Definition 2.2. (Chartrand and Lesniak [1]) The cycle graph C_m is a connected graph that forms a circle with degree of m-vertices are equal to two.

Definition 2.3. (Kaladevi and Kavitha [4]) Let G be a graph with n vertices, $v_1, v_2, \ldots, v_n$, and H is a graph with m vertices. The corona operation of two graphs G and H, $G \odot H$, is defined as the graph obtained by taking one copy of G of order n and n copies of graph H and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.

Lemma 2.1. *(Chartrand et al. [2]) $rc(G) = src(G) = 1$ if and only if G is a complete graph.*

Proposition 2.1. *(Chartrand et al. [2]) Let C_n be a cycle graph. For each integer $n \geq 4$, $rc(C_n) = src(C_n) = \lceil \frac{n}{2} \rceil$.*

A fan graph F_n is a corona graph $K_1 \odot P_n$ where each vertex in P_n is connected to the vertex in K_1. A wheel graph can be considered as a corona product of K_1 with a cycle graph C_n. Thus a wheel graph is $K_1 \odot C_n$.

Proposition 2.2. *(Syafrizal et al. [5]) For $n \geq 2$, the rainbow connection number*

$$rc(F_n) = \begin{cases} 1, & n = 2; \\ 2, & 3 \leq n \leq 6; \\ 3, & n \geq 7. \end{cases}$$

Proposition 2.3. *(Syafrizal et al. [5]) For integers $n \geq 2$, the strong rainbow connection number of the fan*

$$src(F_n) = \begin{cases} 1, & \text{for } n = 2 \\ 2, & \text{for } 3 \leq n \leq 6 \\ \lceil \frac{n}{3} \rceil, & \text{for } n \geq 7 \end{cases}$$

Proposition 2.4. *(Chartrand et al. [2]) For $n \geq 3$, the rainbow connection number of the wheel W_n is*

$$rc(W_n) = \begin{cases} 1, & n = 3; \\ 2, & 4 \leq n \leq 6; \\ 3, & n \geq 7. \end{cases}$$

Proposition 2.5. *(Chartrand et al. [2]) For integers $n \geq 3$, the strong rainbow connection number of the wheel W_n is*

$$src(W_n) = \lceil \frac{n}{3} \rceil$$

3. Main Results

3.1. Corona Graph $C_m \odot P_n$

Let $G = C_m \odot P_n$ then $V(G) = V(C_m) \bigcup V(mP_n)$ where $V(C_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(mP_n) = \{v_1^1, , v_1^2, \ldots, v_1^n\} \bigcup \{v_2^1, v_2^2, \ldots, v_2^n\} \bigcup \cdots \bigcup \{v_m^1, v_m^2, \ldots, v_m^n\}$. $E(G) = \{\{(v_1, v_2), (v_2, v_3), \ldots, (v_m, v_1)\} \bigcup \{(v_1^1, v_1^2), (v_1^2, v_1^3), \ldots, (v_1^{(n-1)}, v_1^n)\} \bigcup \{(v_2^1, v_2^2), (v_2^2, v_2^3), \ldots, (v_2^{(n-1)}, v_2^n)\} \bigcup \ldots \bigcup \{(v_m^1, v_m^2), (v_m^2, v_m^3), \ldots, (v_m^{(n-1)}, v_m^n)\} \bigcup \{(v_1, v_1^1), (v_1, v_1^2), \ldots, (v_1, v_1^n)\} \bigcup \{(v_2, v_2^1), (v_2, v_2^2), \ldots, (v_2, v_2^n)\} \bigcup \ldots \bigcup \{(v_m, v_m^1), (v_m, v_m^2), \ldots, (v_m, v_m^n)\}\}$

Graph $C_m \odot P_n$ is a corona graph where C_m can be considered as its center and every vertex v_i in C_m is connected to every vertex in P_n.

Theorem 3.1. *The rainbow connection number of corona craph $C_m \odot P_n$ is*

$$rc(C_m \odot P_n) = \begin{cases} 4, & \text{for } m = 3, n \geq 2 \\ \lceil \frac{m}{2} \rceil + 3, & \text{for } m > 3, n \geq 2 \end{cases}$$

Proof. Using the similar coloring of C_n as in Chartrand *et al.*[1], and combine it with the coloring for F_n as in Syafrizal *et al.*[5], we find the rc($C_m \odot P_n$) in two cases.

Case 1. For $m = 3, n \geq 2$, $rc(C_3 \odot P_n) = 4$.

Since $C_3 = K_3$, then according to Lemma 2.1, $rc(C_3) = 1$. Based on Proposition 2.2, we have

$$rc(F_n) = \begin{cases} 1, & n = 2; \\ 2, & 3 \leq n \leq 6; \\ 3, & n \geq 7. \end{cases}$$

Suppose $v_a, v_b \in V(C_3 \odot P_n)$ then there are 4 cases to determine $rc(C_3 \odot P_n)$.

- Case 1.1. $v_a, v_b \in V(C_3)$
  Based on Lemma 2.1, we obtained that $rc(V(C_3) = 1$. Then the length of rainbow path $v_a - v_b$ is equal to one.

- Case 1.2. $v_a \in V(C_3), v_b \in V(P_n^i); i = 1, 2, 3$
  Based on Proposition 2.2, since v_i is equal to one of the center vertex v_p, we obtained that the length of rainbow path is equal to r, where
  $r = \begin{cases} 1, & n = 2; \\ 2, & 3 \le n \le 6; \\ 3, & n \ge 7. \end{cases}$

- Case 1.3. $v_a \in V(C_3), v_b \in V(P_n^i); i = 1, 2, 3$. Since v_a is not equal to v_p, so there is a path $v_b - v_p - v_a$, with $v_a \in V(C_3)$, then the length of rainbow path $v_a - v_b$ is equal to two.

- Case 1.4. $v_a \in V(P_n^i), v_b \in V(P_n^j); i, j = 1, 2, 3$
  Based on Definitions 2.1 and Definitions 2.2, it can be concluded that vertices $\{v_1, v_2, v_3\}$ in C_3 is the central of fan graph and its are vertices of cycle (C_3). Thus the vertices in the three fan subgraphs (note that the rc$(F_n) = 3)$ require only additional one color to have a rainbow path. Thus the minimum length of the rainbow path $v_a - v_b$ is equal to four.

From the four cases, it is proved that $rc(C_3 \odot P_n) = 4$.

Case 2. For $m > 3, n \ge 2$, $rc(C_m \odot P_n) = \lceil \frac{m}{2} \rceil + 3$. Based on Proposition 2.1, we obtained that $rc(C_m) = \lceil \frac{m}{2} \rceil$. Based on Proposition 2.2, we know that $rc(F_n) = \begin{cases} 1, & n = 2; \\ 2, & 3 \le n \le 6; \\ 3, & n \ge 7. \end{cases}$ We will show that $rc(C_m \odot P_n) = \lceil \frac{m}{2} \rceil + 3$, for $n \ge 2$.

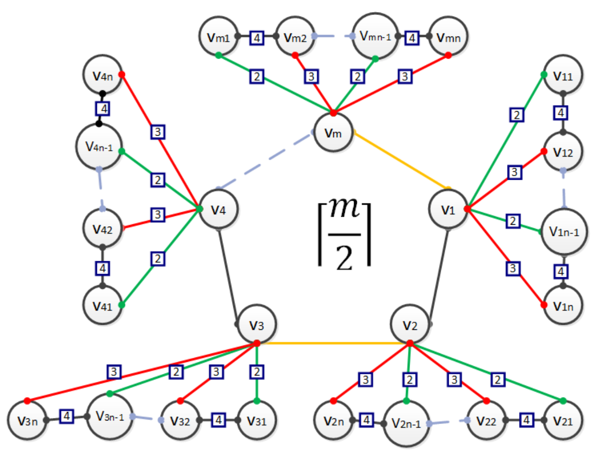

Figure 1. The coloring ilustration of $C_m \odot P_n$ Graph

Figure 1 shows the ilustration of the coloring of graph $C_m \odot P_n$. From Figure 1, it can be seen that:

The vertices of $P_n^i, i = 1, 2, \ldots, m$ which is connected to vertex v_i in the cycle C_m, is formed fan subgraphs of $C_m \odot P_n$. Let $v_a, v_b \in V(C_m \odot P_n)$ then there are four cases to determine $rc(C_m \odot P_n)$.

- Case 2.1. $v_a, v_b \in V(C_m)$
  Based on Proposition 2.1, we obtained that $rc(C_m) = \lceil \frac{m}{2} \rceil$. Thus the minimum length of rainbow path $v_a - v_b$ is equal to $\lceil \frac{m}{2} \rceil$.

- Case 2.2. $v_a \in V(C_m), v_b \in V(P_n^i); i = 1, 2, 3, \ldots, m$
  Based on Proposition 2.2, since v_a is equal to the center vertex v_p for one $p \in \{v_1, \ldots, v_n\}$, and
  $$rc(F_n) = \begin{cases} 1, & n = 2; \\ 2, & 3 \leq n \leq 6; \\ 3, & n \geq 7. \end{cases}$$
  Thus we know that the maximum length of rainbow path of $v_a - v_b$ is equal to $\lceil \frac{m}{2} \rceil$.

- Case 2.3. $v_a \in V(C_m), v_b \in V(P_n^i); i = 1, 2, 3, \ldots, m$
  Since v_a is not equal to v_p, then there is a path $v_b - v_p - \ldots - v_a$, with $v_a \in V(C_m)$, then the length of path $v_a - v_p - \cdots - v_b$ is equal to $\lceil \frac{m}{2} \rceil + 1$.

- Case 2.4. $v_a \in V(P_n^i), v_b \in V(P_n^j); i, j = 1, 2, 3, \ldots, m$.
  Based on Definitions 2.1 and Definitions 2.2, it can be concluded that in $C_m \odot P_n$, the central of fan graphs $\{v_1, v_2, v_3, \ldots, v_m\}$ formed a cycle (C_m). Thus the vertices in m-fan graphs require only three additional colors.

From the four cases, it is proved that $rc(C_m \odot P_n) = \lceil \frac{m}{2} \rceil + 3$.

□

Theorem 3.2. *Strong rainbow connection number of corona graph $C_m \odot P_n$.*

$$src(C_m \odot P_n) = \begin{cases} (\lceil \frac{n}{3} \rceil \cdot 3) + 1, & \textit{for } m = 3, n \geq 2 \\ (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil, & \textit{for } m > 3, n \geq 2 \end{cases}$$

Proof. Using the similar coloring of C_n as in Chartrand *et al.*[1], and combine with the coloring for F_n as in Syafrizal*et al.*[5], we prove the rc of $C_m \odot P_n$ in two parts. The first part is for $m = 3$ and the second part is for the case $m > 3$ as follows.

Case 1 For $m = 3, n \geq 2$, $src(C_m \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$. Based on Lemma 2.1, we obtained that $src(C_3) = 1$. Based on Proposition 2.3, we know that

$$src(F_n) = \begin{cases} 1, & \text{for } n = 2 \\ 2, & \text{for } 3 \leq n \leq 6 \\ \lceil \frac{n}{3} \rceil, & \text{for } n \geq 7 \end{cases}$$

We will show that $src(C_3 \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$, for $m = 3, n \geq 2$.

Graph $C_3 \odot P_n$ has three fan subgraphs whose respective center vertices are $\{v_1, v_2, v_3\}$, and its are the vertices of the cycle C_3. Suppose $v_a, v_b \in V(C_3 \odot P_n)$ then there are four cases to determine $src(C_3 \odot P_n)$

- Case 1.1. $v_a, v_b \in V(C_3)$
  Based on Lemma 2.1, we obtained that $rc(C_3) = 1$.

- Case 1.2. $v_a \in V(C_3), v_b \in V(P_n^i); i = 1, 2, 3$
  Based on Proposition 2.4, since v_a is equal to center vertex v_p, we know that

$$src(F_n) = \begin{cases} 1, & \text{for } n = 2 \\ 2, & \text{for } 3 \leq n \leq 6 \\ \lceil \frac{n}{3} \rceil, & \text{for } n \geq 7 \end{cases}$$

- Case 1.3. $v_a \in V(C_3), v_b \in V(P_n^i); i = 1, 2, 3$ Since v_a is not equal to v_p,then there is a path $v_b - v_p - v_a$, with $v_a \in V(C_3)$. Thus the length of the geodesic path is equal to two.

- Case 1.4. $v_a \in V(P_n^i), v_b \in V(P_n^j); i, j = 1, 2, 3$
  Consider the two fan subgraphs with center vertices v_i and v_j which are connected together. Since the two graphs are connected, then there is one edge connecting the two centers v_i and v_j of the fan so that its geodesic path is three.
  Src is a rainbow connection that requires the number of colors on the edges is calculating for the geodesic path. This causes each of the fan subgraph to have a different color. Based on Lemma 2.3, by the same coloring that one color can only be used at most three times.
  Color the spokes of $P_n^i \neq P_n^j \neq P_n^k$, with the additional color to the connecting edge of each F_n. Thus, we obtained $src(C_3 \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$.

It is proved that $src(C_3 \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$.

Case 2 For $m > 3, n \geq 2$, $src(C_m \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$. Based on Proposition 2.1, we obtained that $src(C_m) = \lceil \frac{m}{2} \rceil$. Based on Proposition 2.4, we know that

$$src(F_n) = \begin{cases} 1, & \text{for } n = 2 \\ 2, & \text{for } 3 \leq n \leq 6 \\ \lceil \frac{n}{3} \rceil, & \text{for } n \geq 7 \end{cases}$$

We will show that $src(C_m \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil, \text{for} n \geq 2$.

From the general form (ilustrated in Figure 2), we can see that:
$C_m \odot P_n$ has m fan subgraphs which its respective center vertices are $\{v_1, v_2, v_3, \ldots, v_m\}$. Its are vertices of the cycle C_m. Suppose that $v_a, v_b \in V(C_m \odot P_n)$ then there are four cases to find $src(C_m \odot P_n)$.

- Case 2.1. $v_a, v_b \in V(C_m)$
  Based on Proposition 2.1, we knew that $src(C_m) = \lceil \frac{m}{2} \rceil$. Thus the length of the geodesic path from $v_a - v_b$ is equal to $\lceil \frac{m}{2} \rceil$

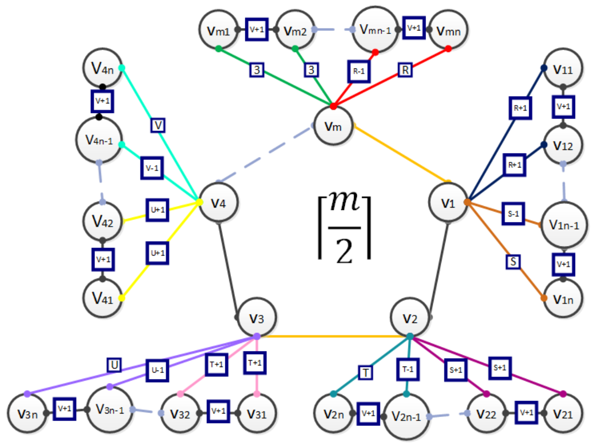

Figure 2. src $C_m \odot P_n$ Graph

- Case 2.2. $v_a \in V(C_m), v_b \in V(P_n^i); i = 1, 2, 3, \ldots, m$
  Based on Proposition 2.3, since v_a equal to its center vertex v_p, we obtained that the length of the geodesic path is equal to

$$src(F_n) = \begin{cases} 1, & \text{for } n = 2 \\ 2, & \text{for } 3 \leq n \leq 6 \\ \lceil \frac{n}{3} \rceil, & \text{for } n \geq 7 \end{cases}$$

- Case 2.3. $v_a \in V(C_m), v_b \in V(P_n^i); i = 1, 2, 3, \ldots, m$
  Since v_a is not equal to v_p, then there is a path $v_b - v_p - \ldots - v_a$, with $v_a \in V(C_m)$. Thus the length of the geodesic path is equal to $\lceil \frac{m}{2} \rceil + 1$.

- Case 2.4. $v_a \in V(P_n^i), v_b \in V(P_n^j); i, j = 1, 2, 3, \ldots, m$
  Consider the two fan subgraphs which their center is adjacent. Then there is only one edge connecting these two center v_i and v_{i+1} of the two fans so that its geodesic path is maximum three.
  Finding the strong rainbow coloring number for $C_m \odot P_n$ needs to guarantee that every pair of vertices where the verex come from the different fan subgraph, conncet with the rainbow geodes path. This causes each fan subgraph having different color. Based on Lemma 2.3, and using the same coloring as in the proof of Lemma 2.3, we know that one color can only be used at most three times.

  Color the rim of $P_n^i \neq P_n^j \neq P_n^k$, with the additional one color. Then the length of the geodesic path is equal to $(\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$.

Based on the four cases, it is proved that the $src(C_m \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$.

□

Theorem 3.3. *Rainbow connection cumber of corona graph $C_m \odot C_n$.*

$$rc(C_m \odot C_n) = \begin{cases} 4, & \textit{for } m = 3, n \geq 3 \\ \lceil \frac{m}{2} \rceil + 3, & \textit{for } m > 3, n \geq 3 \end{cases}$$

Proof. Using the similar coloring of C_n as in Chartrand *et al.*[1],, which is ilustrated in Figure 3 and Figure 4, we divide the proof in two cases.

Case 1. For $m = 3, n \geq 3$, $rc(C_3 \odot C_n) = 4$.

Since $C_3 = K_3$ then according to Lemma 2.1, $rc(C_3) = 1$. Based on Proposition 2.4, we have $rc(W_n) = \begin{cases} 1, & n = 3; \\ 2, & 4 \leq n \leq 6; \\ 3, & n \geq 7. \end{cases}$ It will be shown that $rc(C_3 \odot C_n) = 4$, for $n \geq 3$.

Consider the coloring as ilustrated in Figure 3.

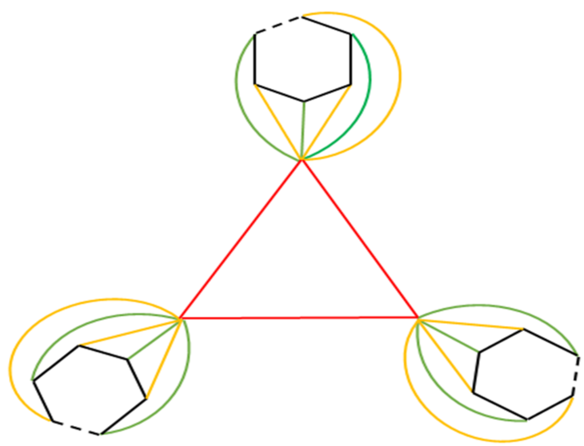

Figure 3. Coloring ilustration of $C_3 \odot C_n$ Graph

From the Figure 3, it can be seen that the graph $C_3 \odot C_n$ has three wheel subgraphs, where the respective center vertices $\{v_1, v_2, v_3\}$ are vertices of a cycle C_3. Suppose that $v_a, v_b \in V(C_3 \odot C_n)$ then there are four cases to considered for finding the $rc(C_3 \odot C_n)$

- Case 1.1. $v_a, v_b \in V(C_3)$
  Based on Lemma 2.1, we knew that $rc(C_3) = 1$. Thus the length of the rainbow path $v_a - v_b$ is equal to one.

- Case 1.2. $v_a \in V(C_3), v_b \in V(C_n^i); i = 1, 2, 3.$
  Based on Proposition 2.4, since v_a is equal to v_p (center vertex of one of a wheel subgraph), we obtained that the length of the rainbow path of $v_a - v_b$ is equal to r, where
  $r = \begin{cases} 1, & n = 3; \\ 2, & 4 \leq n \leq 6; \\ 3, & n \geq 7. \end{cases}$

- Case 1.3. $v_a \in V(C_3), v_b \in V(C_n^i); i = 1, 2, 3$. Since v_a is not equal to the center vertex v_p, then there is a path $v_b - v_p - v_a$, with $v_a \in V(C_3)$. Thus the length of the rainbow path is equal to two.

- Case 1.4. $v_a \in V(C_n^i), v_b \in V(C_n^j); i, j = 1, 2, 3$
  Based on Definitions 2.3, it can be concluded that C_3 is the central cycle and vertices $\{v_1, v_2, v_3\}$ are vertices of (C_3). Thus the three wheel graphs require only one additional color.

It is proved that $rc(C_3 \odot C_n) = 4$.

Case 2. For $m > 3, n \geq 3, rc(C_m \odot C_n) = \lceil \frac{m}{2} \rceil + 3$.
Based on Proposition 2.1, we knew that $rc(C_m) = \lceil \frac{m}{2} \rceil$. Based on Proposition 2.4, we have $rc(W_n) = \begin{cases} 1, & n = 3; \\ 2, & 4 \leq n \leq 6; \\ 3, & n \geq 7. \end{cases}$ We will show that $rc(C_m \odot C_n) = \lceil \frac{m}{2} \rceil + 3, for n \geq 3$.

Figure 4 gives the ilustration of coloring of $C_m \odot C_n$ graph.

From the Figure 4, it can be seen that $C_m \odot C_n$ has m wheel subgraphs, where its respective center vertices $\{v_1, v_2, \ldots, v_m\}$ are the vertices of the center cycle C_m. Suppose that $v_a, v_b \in V(C_m \odot C_n)$, then there are four cases to determine $rc(C_m \odot C_n)$

- Case 2.1. $v_a, v_b \in V(C_m)$
  Based on Proposition 2.1, we obtained that $rc(C_m) = \lceil \frac{m}{2} \rceil$

- Case 2.2. $v_a \in V(C_m), v_b \in V(P_n^i); i = 1, 2, 3, \ldots, m$
  Based on Proposition 2.4, since v_a is equal to v_p (one of the verices in the center cycle), we obtained that
  the length of the rainbow path is equal to $\begin{cases} 1, & n = 3; \\ 2, & 4 \leq n \leq 6; \\ 3, & n \geq 7. \end{cases}$

- Case 2.3. $v_a \in V(C_m), v_b \in V(C_n^i); i = 1, 2, 3, \ldots, m$
  Since v_a is not equal to v_p, so there is a path $v_b - v_p - \ldots - v_a$, with $v_a \in V(C_m)$, then the length of the rainbow path is maximum $\lceil \frac{m}{2} \rceil + 1$.

- Case 2.4. $v_a \in V(C_n^i), v_b \in V(C_n^j); i, j = 1, 2, 3, \ldots, m$
  Based on Definitions 2.1 and Definitions 2.2, it can be concluded that $C_m \odot P_n$ has the central vertices, $\{v_1, v_2, \ldots, v_m\}$, of the fan subgraph graph, which they are the vertices of the cycle (C_m). Thus the m-fan subgraphs, where each has $rc = \lceil \frac{m}{2} \rceil$, require only three additional colors.

The four cases proved that $rc(C_m \odot C_n) = \lceil \frac{m}{2} \rceil + 3$.

□

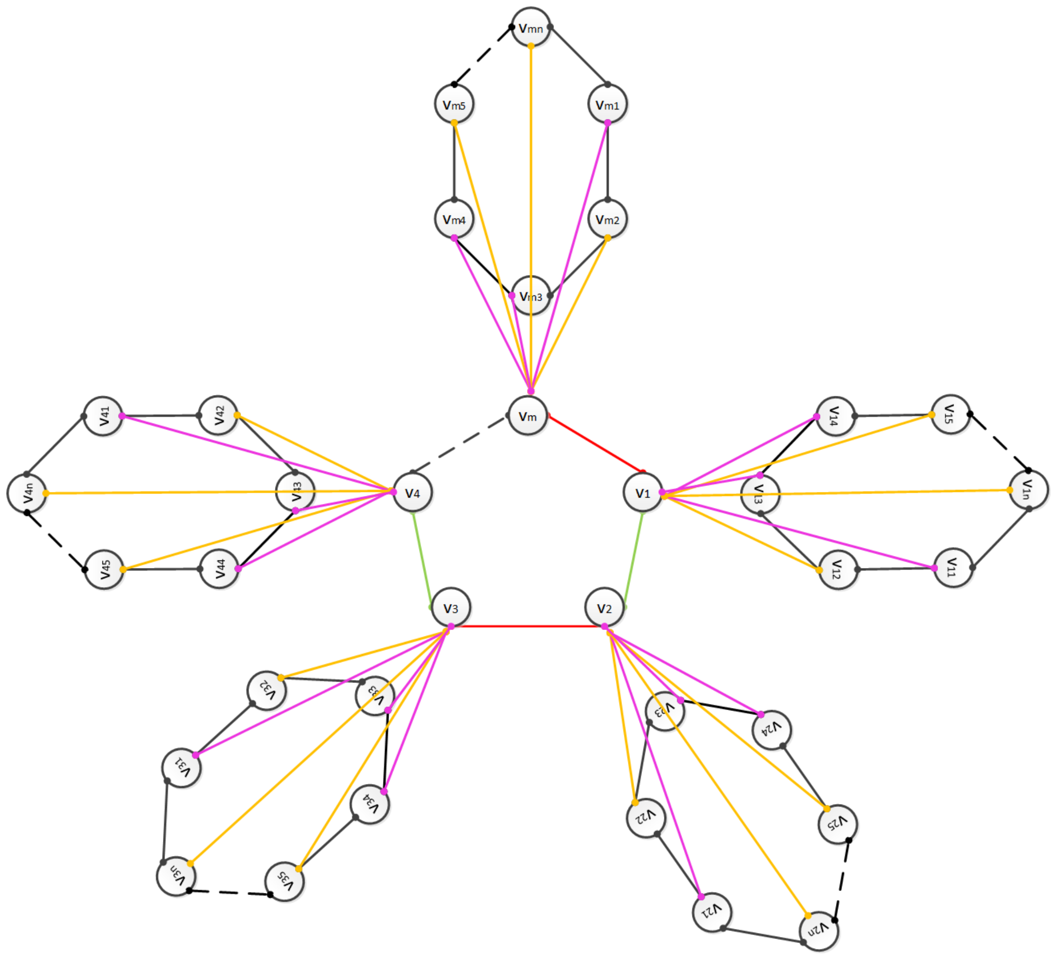

Figure 4. Coloring ilustration of $C_m \odot C_n$

Theorem 3.4. *Strong rainbow connection number of corona graph $C_m \odot C_n$.*

$$src(C_m \odot C_n) = \begin{cases} (\lceil \frac{n}{3} \rceil \cdot 3) + 1, & \textit{for } m = 3, n \geq 3 \\ (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil, & \textit{for } m > 3, n \geq 3 \end{cases}$$

Proof. Similar with the previous theorems, the proof will be divided into two cases.

Case 1. For $m = 3, n \geq 3$, $src(C_m \odot P_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$. Based on Lemma 2.1, we knew that $src(C_3) = 1$. Based on Proposition 2.5, we have

$$src(W_n) = \lceil \frac{n}{3} \rceil$$

We will show that $src(C_3 \odot C_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$, for $m = 3, n \geq 3$.

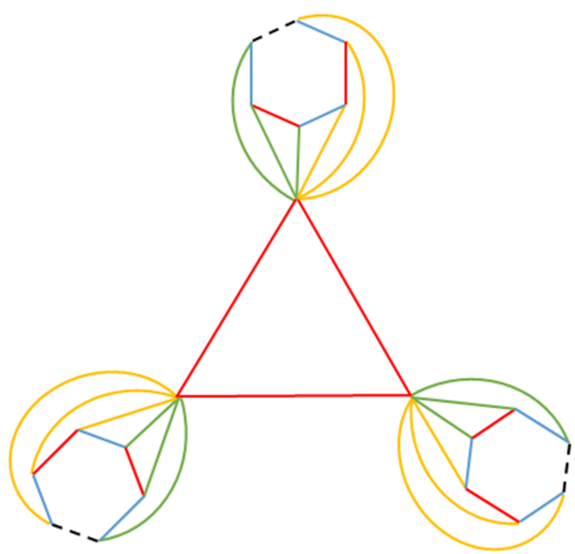

Figure 5. Coloring ilustration of $C_3 \odot C_n$

From Figure 5, it can be seen that $C_3 \odot C_n$ has three wheel subgraphs where their vertex centers $\{v_1, v_2, v_3\}$ are the vertices of C_3. Suppose that $v_a, v_b \in V(C_3 \odot C_n)$, then there are four cases to determine $src(C_3 \odot C_n)$

- Case 1.1. $v_a, v_b \in V(C_3)$
  Based on Lemma 2.1, we knew that $rc(C_3) = 1$. Thus the geodesic path has length one.
- Case 1.2. $v_a \in V(C_3), v_b \in V(C_n^i); i = 1, 2, 3$
  Based on Proposition 2.5, since v_a is equal to one of the vertex center v_p, we obtained that the length of geodesic path from v_a to v_b is equal to $\lceil \frac{n}{3} \rceil$

- Case 1.3. $v_a \in V(C_3), v_b \in V(C_n^i); i = 1, 2, 3$. Since v_a is not equal to the vertex center $v_p \in \{v_i, v_2, v_3\}$, then there is a path $v_b - v_p - v_a$, with $v_a \in V(C_3)$. Thus the length of the geodesic path is equal to three.

- Case 1.4. $v_a \in V(C_n^i), v_b \in V(C_n^j); i, j = 1, 2, 3$
  Look at the two wheel subgraphs, which has center vertex v_i and v_{i+1}. These two center vertices are adjacent. Then the two wheel subgraphs are also connected and the length of the geodesic graph is equal to three.
  The rainbow geodesic path requires the number of colors are equal to its length. This fact make the color of each wheel has to be different. Based on Lemma 2.3, one color can only be used at most three times in each wheel subgraph.
  Thus, we will have the length of geodesic rainbow path is $(\lceil \frac{n}{3} \rceil \cdot 3) + 1$.

From the 4 cases, we prove that $src(C_3 \odot C_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + 1$.

Case 2. For $m > 3, n \geq 3$, $src(C_m \odot C_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$. Based on Proposition 2.1, we obtained $src(C_m) = \lceil \frac{m}{2} \rceil$. Based on Proposition 2.5,

$$src(W_n) = \lceil \frac{n}{3} \rceil$$

We will show that $src(C_m \odot C_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$, for $n \geq 3$.

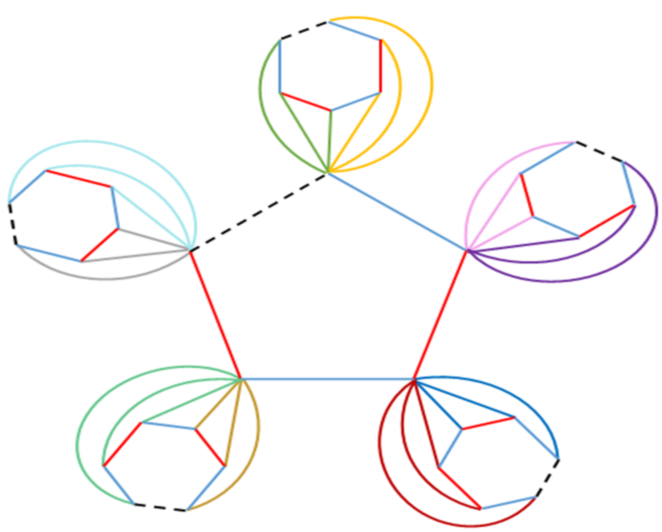

Figure 6. Coloring ilustration of $C_m \odot C_n$

From the general form which can be seen in Figure 6, that we have $C_m \odot C_n$ has m wheel subgraphs. The wheel subgraphs have its center vertices $\{v_1, v_2, \ldots, v_m\}$ which are vertices of C_m. Suppose that $v_a, v_b \in V(C_m \odot C_n)$ then there are four cases to determine $src(C_m \odot C_n)$

- Case 2.1. $v_a, v_b \in V(C_m)$
  Based on Proposition 2.1, we knew that $rc(C_m) = \lceil \frac{m}{2} \rceil$. Thus the length of the geodesic path is $\lceil \frac{m}{2} \rceil$.

- Case 2.2. $v_a \in V(C_m), v_b \in V(C_n^i); i = 1, 2, 3, \ldots, m$
  Based on Proposition 2.5, since v_a is equal to v_p (one of the center vertex), we knew that $src(W_n) = \lceil \frac{n}{3} \rceil$, which is the same with the length of geodesic path from v_a to v_b.

- Case 2.3. $v_a \in V(C_m), v_b \in V(C_n^i); i = 1, 2, 3, \ldots, m$
  Since v_a is not equal to one of the center vertex v_p, then there is a path $v_b - v_p - \ldots - v_a$, with $v_a \in V(C_m)$. Thus the length of the geodesic path is $\lceil \frac{m}{2} \rceil + 1$.

- Case 2.4. $v_a \in V(C_n^i), v_b \in V(C_n^j); i, j = 1, 2, 3, \ldots, m$
  Consider the two wheel subgraphs with center vertices are adjacent, which are v_i and $v_i + 1$, respectively. Thus the length of the geodesic rainbow path is equal to three. Based on Lemma 2.3, the same color only can be used three times in the spoke of the wheel subgraphs. Color the rim of the wheels with the aditional color to connect the vertices in outher cyccle of the wheel.Thus we obtained the length of the geodesic rainbow path is equal to $= (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$.

From the four cases, we proved that $src(C_m \odot C_n) = (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil$.

□

4. Conclusion

From the discussion that we have in this paper, it can be concluded that

1. Rainbow connection number for corona graph $C_m \odot P_n$ and $C_m \odot C_n$.

$$rc(C_m \odot P_n) = \begin{cases} 4, & \text{for } m = 3, n \geq 2 \\ \lceil \frac{m}{2} \rceil + 3, & \text{for } m > 3, n \geq 2 \end{cases}$$

$$rc(C_m \odot C_n) = \begin{cases} 4, & \text{for } m = 3, n \geq 3 \\ \lceil \frac{m}{2} \rceil + 3, & \text{for } m > 3, n \geq 3 \end{cases}$$

2. Strong rainbow connection number for corona graph $C_m \odot P_n$ and $C_m \odot C_n$.

$$src(C_m \odot P_n) = \begin{cases} (\lceil \frac{n}{3} \rceil \cdot 3) + 1, & \text{for } m = 3, n \geq 2 \\ (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil, & \text{for } m > 3, n \geq 2 \end{cases}$$

$$src(C_m \odot C_n) = \begin{cases} (\lceil \frac{n}{3} \rceil \cdot 3) + 1, & \text{for } m = 3, n \geq 3 \\ (\lceil \frac{n}{3} \rceil \cdot 3) + \lceil \frac{m}{2} \rceil, & \text{for } m > 3, n \geq 3 \end{cases}$$

3. $C_m \odot C_n$ has the same rc and src as $C_m \odot P_n$, since both of the graphs have the same characteristic where there is a center vertex v_p in wheel graph and fan graph. Thus, they have the same coloring type for rainbow coloring.

Acknowledgement

This research is supported by PITTA UI 2017 research grant from Universitas Indonesia.

References

[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Third Edition, Chapman and Hall/CRC (1996).

[2] G. Chartrand, G. L. Johns, K. A. Mc Keon, and P. Zhang, Rainbow connection in graphs, *Math. Bohem.* **133** (2008), 85–98.

[3] X. Li and Y. Sun, *Rainbow Connections of Graphs*, Springer, New York. 2012.

[4] V. Kaladevi and G. Kavitha, Edge-odd graceful labeling of some corona graphs, *Proc. ICMEB.* (2012)

[5] Sy. Syafrizal, G. H. Medika, and L. Yulianti, The rainbow connection of fan and sun, *Appl. Math. Sci.* **7** (2013), 3155–3160.