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On star coloring of Mycielskians

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Abstract

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed a graph transformation that transforms a graph G into a new graph $\mu(G)$, we now call the Mycielskian of G, which has the same clique number as G and whose chromatic number equals $\chi(G)+1$. In this paper, we find the star chromatic number for the Mycielskian graph of complete graphs, paths, cycles and complete bipartite graphs.

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1. Introduction

The notion of star chromatic number was introduced by Branko Grünbaum in 1973. A star coloring [1, 4, 5] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. Equivalently, in a star coloring, the induced subgraphs formed by the vertices of any two color classes has connected components that are star graphs. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G.

Guillaume Fertin et al.[5] gave the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outerplanar graphs, and 2-dimensional grids. They also investigated and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, d-dimensional grids ($d \geq 3$), d-dimensional tori ($d \geq 2$), graphs with bounded treewidth, and cubic graphs.

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Albertson et al.[1] showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, even when G is a graph that is both planar and bipartite. The problems of finding an optimal star colorings is NP-hard and remain so even for bipartite graphs.

Preliminaries

We consider only finite, undirected, loopless graphs without multiple edges. The open neighborhood of a vertex x in a graph G, denoted by $N_G(x)$, is the set of all vertices of G, which are adjacent to x. Also, $N_G[x] = N_G(x) \cup \{x\}$ is called the closed neighborhood of x in the graph G.

In this paper, by G we mean a connected graph. From a graph G, by Mycielski's construction [3, 7, 8], we get the *Mycielskian* $\mu(G)$ of G with $V(\mu(G)) = V \cup U \cup \{z\}$, where

$$V = V(G) = \{x_1, \ldots, x_n\}, \quad U = \{y_1, \ldots, y_n\}, \text{ and}$$
$$E(\mu(G)) = E(G) \cup \{y_i x : x \in N_G(x_i) \cup \{z\}, i = 1, \ldots, n\}.$$

A *star coloring* of a graph G is a proper coloring of G such that no path of length 3 in G is bicolored. The *star chromatic number* of a graph G is the minimum number of colors which are necessary to star color G. It is denoted by $\chi_s(G)$ for star coloring.

Additional graph theory terminology used in this paper can be found in [2, 6].

In order to prove our results, we shall use the following generalities and theorems by Guillaume et al. [5].

Proposition 1.1. *[5] For any graph G of order n and size m, $\chi_s(G) \geq \dfrac{2n+1-\sqrt{\Delta}}{2}$, where $\Delta = 4n(n-1) - 8m + 1$.*

Proposition 1.2. *[5] Let T be a tree and V_1 and V_2 be the bipartition of its set of vertices, then there exists a star coloring of T $c : V(T) \rightarrow \{0, 1, 2, 3\}$ such that if $v \in V_1$ then $c(v) \in \{0, 2\}$ and if $v \in V_2$ then $c(v) \in \{1, 3\}$.*

Corollary 1.1. *[5] If G is a planar graph with girth $g \geq 5$, then $\chi_s(G) \leq 32$. If G is a planar graph with girth $g \geq 7$, then $\chi_s(G) \leq 12$.*

Observation 1.1. [5] For any graph G and for any $1 \leq \alpha \leq |V(G)|$, let $G_1, \ldots, G_p$ be the p connected components obtained by removing α vertices from G. In that case, $\chi_s \leq \max_i\{\chi_s(G_i)\} + \alpha$.

Remark 1.1. [5] For any $\alpha \geq 1$, the above result is optimal for complete bipartitie graphs $K_{n,m}$. Without loss of generality, suppose $n \leq m$ and let $\alpha = n$. Remove the $\alpha = n$ vertices of partition V_n. We then get m isolated vertices, which can be independently colored with a single color. Then, give a unique color to the $\alpha = n$ vertices. We then get a star coloring with $n + 1$ colors ; this coloring can be shown to be optimal by theorem 1.2.

Observation 1.2. [5] For any graph G that can be partitioned into p stables $S_1, \ldots, S_p$, $\chi_s(G) \leq 1 + |V(G)| - \max_i\{|S_i|\}$.

Theorem 1.1. *[5] If C_n is a cycle with $n \geq 3$ vertices, then*

$$\chi_s(C_n) = \begin{cases} 4 & \textit{when } \ n = 5 \\ 3 & \textit{otherwise.} \end{cases}$$

Theorem 1.2. *[5] Let $K_{n,m}$ be a complete bipartite graph with $n + m$ vertices. Then $\chi_s(K_{n,m}) = \min\{m, n\} + 1$.*

In the following section, we prove results concerning the star chromatic number of Mycielskian graph of complete graphs, paths, cycles and complete bipartite graphs.

First, we define the vertex sets as follows,

$$V\left(K_n\right) = V\left(P_n\right) = V\left(C_n\right) = \{u_i : 1 \leq i \leq n\}$$

$$V\left(\mu\left(K_n\right)\right) = V\left(\mu\left(P_n\right)\right) = V\left(\mu\left(C_n\right)\right) = \{u_i : 1 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq n\} \cup \{z\}$$

$$V\left(K_{m,n}\right) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$$

$$\begin{aligned} V\left(\mu\left(K_{m,n}\right)\right) \quad = \quad & \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\} \cup \{u'_i : 1 \leq i \leq m\} \\ & \cup \left\{v'_j : 1 \leq j \leq n\right\} \cup \{z\} \end{aligned}$$

2. Results

Star Coloring of Mycielskian of Complete Graphs

Theorem 2.1. *For $n \geq 2$, $\chi_s(\mu(K_n)) = n + 2$.*

Proof. Let σ be a mapping from $V\left(\mu\left(K_n\right)\right)$ defined as follows: $\sigma\left(u_i\right) = i : 1 \leq i \leq n$, $\sigma(u'_i) = n + 1 : 1 \leq i \leq n$ and $\sigma(z) = n + 2$. Thus $\chi_s\left(\mu\left(K_n\right)\right) \leq n + 2$. First note that at least n colors are needed to assign for vertices $u_i : 1 \leq i \leq n$, since the subgraph induced by these n vertices is isomorphic to K_n. However, n colors are not enough to star color $\mu\left(K_n\right)$, because if only n colors are allowed then $\sigma\left(u'_i\right) = i : 1 \leq i \leq n$ and this case will not satisfy a proper star coloring. Thus $\chi_s\left(\mu\left(K_n\right)\right) \geq n + 1$.

Now suppose that $n + 1$ colors are allowed. If $\sigma\left(u'_i\right) = n + 1 : 1 \leq i \leq n$. Then, z has received any one color from $u_i = i : 1 \leq i \leq n$ and z is adjacent to u'_i for every $1 \leq i \leq n$. Thus, $n + 1$ colors do not suffice to star color $\mu\left(K_n\right)$ and consequently $\chi_s\left(\mu\left(K_n\right)\right) \geq n + 2$. Therefore, $\chi_s\left(\mu\left(K_n\right)\right) = n + 2$. □

Star Coloring of Mycielskian of Paths

Theorem 2.2. *For any positive integer $n > 3$, $\chi_s(\mu(P_n)) = 5$.*

Proof. Let σ be a mapping from $V(\mu(P_n))$ defined as follows: $\sigma\left(u_i\right) = i \bmod 3$; $\sigma(u_i') = 3 : 1 \leq i \leq n$ and $\sigma(z) = 4$. Thus, $\chi_s\left(\mu\left(P_n\right)\right) \leq 5$. Color the vertices of $u_i : 1 \leq i \leq n$ alternatively by colors 1,2 and 0. Thus for any vertex u_i, its two neighbours are assigned distinct colors and consequently this is a valid star coloring. First, note that the cycle of length 5, C_5 is a subgraph of $\mu(P_n)$. Thus, $\chi_s(\mu(P_n)) \geq 4$, by Theorem 1.1. Suppose, only 4 colors are used in $\mu\left(P_n\right)$.

In this case, $u_i : 1 \leq i \leq n$ and $u_i' : 1 \leq i \leq n$ can be assigned colors 0, 1, 2 and 3. If $\sigma\left(u_i\right) = i \bmod 4$ and the vertices $u_i' : 1 \leq i \leq n$ has received the colors 1, 0, 3 and 2 alternatively, none of these colors can be given to z. If $\sigma(u_i) = i + 1 \bmod 4$ and the vertices $u_i' : 1 \leq i \leq n$ has received the colors $0, 2, 3$ and 1 alternatively, none of these colors can be given to z. If $\sigma\left(u_i\right) = i \bmod 3$ and $\sigma\left(u_i'\right) = 4, 1 \leq i \leq n$, none of these colors can be given to z. Therefore $\mu\left(P_n\right)$ must be colored with at least 5 different colors. Thus, $\chi_s(\mu(P_n)) \geq 5$ and hence, $\chi_s(\mu(P_n)) = 5$. □

Star Coloring of Mycielskian of Cycles

Theorem 2.3. *For any positive integer n,*

$$\chi_s(\mu(C_n)) = \begin{cases} 5 & \textit{if } n = 3k \textit{ and } n = 3k+2 \\ 6 & \textit{if } n = 3k+1 \end{cases}$$

where k is a positive integer.

Proof. Let σ be a mapping from $V(\mu(C_n))$ defined as follows:
Case 1. For $n = 3k$, $\sigma\left(u_i\right) = i \bmod 3$, $1 \leq i \leq n$; $\sigma\left(u_i'\right) = 3$, $1 \leq i \leq n$ and $\sigma\left(z\right) = 4$. Thus, $\chi_s(\mu(C_n)) \leq 5$. Clearly, at least 3 colors are needed to assign to vertices $u_i : 1 \leq i \leq n$. First, color alternatively the vertices around the cycle by colors 1,2 and 0. Thus, for any vertex u_i, its two neighbors are assigned distinct colors and consequently this is a valid star coloring. However, 3 colors are not enough to star color $\mu\left(C_n\right)$, because if only 3 colors are allowed then for $1 \leq i \leq n$, $\sigma\left(u_i'\right) = i \mod 3$ and this case will not satisfy a proper star coloring. Thus $\chi_s\left(\mu\left(C_n\right)\right) \geq 4$. From Theorem 2.2, it follows that $\chi_s(\mu(C_n)) \geq 5$. Hence, $\chi_s(\mu(C_n)) = 5$.
Case 2. For $n = 3k+1$, $\sigma\left(u_i\right) = i \bmod 3$, $1 \leq i \leq n-1$; $\sigma\left(u_n\right) = 2$; $\sigma\left(z\right) = 5$ and

$$\sigma(u_i') = \begin{cases} 3 & \text{if } i \equiv 0 \bmod 3 \\ 4 & otherwise. \end{cases}$$

Color the vertices $u_i : 1 \leq i \leq n-1$ of $\mu\left(C_n\right)$ consecutively, by repeating the sequence of colors 1, 2 and 0. There remain one uncolored vertex, to which assign color 2. Note that the cycle of length 5, C_5 is a subgraph of $\mu(C_n)$. It can be easily checked that $\chi_s(\mu(C_n)) = 4$ and thus $\chi_s(\mu(C_n)) \geq 4$. However, 4 colors are not enough to star color $\mu\left(C_n\right)$, because if only 4 colors are allowed, then $\sigma\left(u_i'\right) = \sigma\left(u_i\right)$, $i : 1 \leq i \leq n$ which results in bicolored path $\left\{u_n' u_{n-1} u_n u_{n-1}'\right\}$ and a proper star coloring is not satisfied. Thus $\chi_s\left(\mu\left(C_n\right)\right) \geq 5$.

Now suppose that 5 colors are allowed. If

$$\sigma(u_i') = \begin{cases} 3 & \text{if } i \equiv 0 \bmod 3 \\ 4 & otherwise. \end{cases}$$

Then $\sigma\left(z\right)$ has received any one color from $u_i = i : 1 \leq i \leq n$. Thus, 5 colors do not suffice to star color $\mu\left(C_n\right)$ and consequently $\chi_s\left(\mu\left(C_n\right)\right) \geq 6$. Therefore $\chi_s\left(\mu\left(C_n\right)\right) = 6$.
Case 3. $n \equiv 2 \bmod 3$
Case 3.1. $n = 5$ or $n = 8$

Star coloring of $\mu(C_5) = 6$ and $\mu(C_8) = 5$ is given in Figure 1 a) and b) respectively.

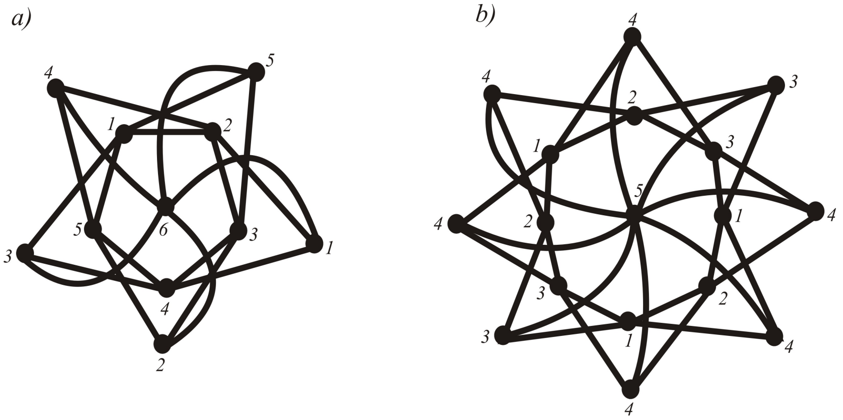

Figure 1. a) $\mu(C_5)$; b) $\mu(C_8)$ with their star coloring.

Case 3.2. $n \geq 11$

Let $n = 8 + 3t$, $t \geq 1$. For $1 \leq i \leq 8$, color the vertices v_i as in Figure 1 b). Then, for $9 \leq i \leq n$ the remaining vertices of $\mu(C_n)$ are colored in the following way,

$$\sigma(u_i) = \begin{cases} 1 & \text{if } \ i \equiv 0 \bmod 3 \\ 2 & \text{if } \ i \equiv 1 \bmod 3 \\ 3 & \text{if } \ i \equiv 2 \bmod 3 \end{cases}$$

and

$$\sigma(u_i') = \begin{cases} 3 & \text{if } \ \sigma(u_i) = 3 \\ 4 & \text{otherwise.} \end{cases}$$

and $\sigma(z) = 5$. Similarly as it was in Case 1, it can be easily checked that σ is proper star 5-coloring. Hence, $\chi_s(\mu(C_n)) = 5$. □

Star Coloring of Mycielskian of Complete Bipartite Graphs

Theorem 2.4. *Let n and m be positive integers, then*

$$\chi_s(\mu(K_{m,n})) = 2(min\left\{m, n\right\} + 1).$$

Proof. Let $m \leq n$. Let σ be a mapping from $V(\mu(K_{m,n}))$ defined as follows. $\sigma(u_i) = i, 1 \leq i \leq m$; $\sigma(v_i) = m + 1, 1 \leq i \leq n$; $\sigma(v_i') = m + 1, 1 \leq i \leq n$; $\sigma(u_i') = m + 1 + i, 1 \leq i \leq m$ and

$\sigma(z) = 2m + 2$. Thus $\chi_s(\mu(K_{m,n})) \leq= 2m + 2$. Now prove that $\chi_s(\mu(K_{m,n})) \geq 2m + 2$. Let S_m and S'_m (resp. S_n and S'_n) be the set of colors used to color the vertices of U_m and U'_m (resp. V_n and V'_n).
Case 1. Consider the vertices U_m (resp. V_n). By Theorem 1.2, $\chi_s(K_{m,n}) \geq m + 1$. Then $\chi_s(\mu(K_{m,n})) \geq m + 1$.
Case 2. Now consider the vertices U_m and U'_m (resp. V_n and V'_n). Any coloring with $2m$ colors will give at least one bicolored cycle of length 4. In that case, there exists at least 2 vertices u_m and u'_m in U_m and U'_m (resp. v_n and v'_n in V_n and V'_n). Since there exists a path of length 4 going through the vertices $\{u'_m v_n\, u_m v'_n\}$ and this path is bicolored with color 1 and 2. Thus, $\chi_s(\mu(K_{m,n})) \geq 2m+1$.
Case 3. Let $V(\mu(K_{m,n})) = U_m \cup U'_m \cup V_n \cup V'_n \cup z$. The vertex z has received any one color from U_m. In that case, there exists a path of length 4 going through the vertices $\{u_m v'_n z v'_n\}$ and this path is bicolored with color 1 and 2. Thus, no coloring that uses $2m + 1$ colors can be a star coloring, and $\chi_s(\mu(K_{m,n})) \geq 2m + 2$.

Therefore $\chi_s(\mu(K_{m,n})) = 2m + 2$. □

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