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Prime and odd prime labelings of several graph classes

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Abstract

A prime labeling of a graph is an assignment of distinct integers to its vertices such that any two adjacent vertices are labeled with numbers that are relatively prime. An odd prime labeling is a variant in which the labels are restricted to odd integers while still preserving the coprimality condition. Motivated by the odd prime graph conjecture, which asserts that every prime graph is odd prime, we study prime and odd prime labelings on various classes of graphs. We construct explicit labeling functions and prove that comb graphs, disjoint unions of comb graphs, triangular book graphs, and modified triangular book graphs $K_{1,1,n} \odot 2S_m$ admit both prime and odd prime labelings. We further establish an odd prime labeling for torch graphs, extending a known prime labeling result for this class. Consequently, all graph families considered in this paper satisfy the odd prime graph conjecture. These results expand the collection of graphs known to admit odd prime labelings and provide additional evidence supporting the conjecture.

Keywords: comb graph, odd prime labeling, prime labeling, torch graph, triangular book graph.
Mathematics Subject Classification : 05C78

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1. Introduction

Graph labeling is a fast-developing branch of graph theory that has been actively studied for many decades and continues to attract significant research interest. In general, graph labeling is a function that assigns integers to vertices, edges, or both, subject to certain prescribed conditions [1]. This topic has received considerable attention due to its rich theoretical structure, which connects graph theory and number theory, as well as its wide range of applications, including communication systems [2], frequency assignment problems [3], coding theory [4], cryptography [5], computer science [6], among others. Numerous labeling variants have been introduced and extensively studied, such as harmonious labeling [7], graceful labeling [8], magic labeling [9], and prime labeling [10], together with their various generalizations. Among these labeling schemes, prime labeling has received particular attention due to its intrinsic connection with divisibility properties and relatively prime integers.

Prime labeling was first introduced by Tout et al. (1982) and has since become an important area of study in graph labeling theory. Let $G = (V, E)$ be a simple undirected graph of order n. A graph G admits a prime labeling if there exists a bijection $f : V(G) \to \{1, 2, \dots, n\}$ such that $\gcd(f(u), f(v)) = 1$ for every edge $uv \in E(G)$. Graph classes known to admit prime labelings include stars, paths, caterpillars with maximum degree at most five [10], helm graphs, fan graphs, flower graphs [11], book graphs [12], olive trees, spiders [13], trees of order up to 50 [14], cycle–star combinations [15], banana trees, palm trees, spider-type graphs, binomial trees [16], generalized Petersen graphs [17], ladder graphs [18], and torch graphs [19].

To accommodate restrictions on the allowable label set, the concept of prime labeling has been generalized to odd prime labeling, in which vertex labels are selected exclusively from the set of odd integers. More precisely, a graph $G = (V, E)$ of order n is said to admit an odd prime labeling if there exists a bijection $f : V(G) \to \{1, 3, 5, \dots, 2n-1\}$ such that adjacent vertices receive labels that are relatively prime. A widely studied conjecture in this area states that every graph admitting a prime labeling also admits an odd prime labeling [20]. Considerable supporting results have been established for many graph classes, such as paths, ladders, fans, complete bipartite graphs, wheels, flowers, cycles, gear graphs, helm, closed helm graphs, generalized Petersen graphs [20], disjoint unions of paths [21], snake, spider, caterpillars of bounded degree, binary trees, prism graphs $GP(n, 1)$, firecracker graphs [22], circular ladders [23], cycle-containing graphs [24], and broom graphs [25]. However, the conjecture remains open in the general case.

In this paper, we study prime and odd prime labelings on several structured graph families, namely comb graphs and their variants, triangular book graphs and their generalizations, and torch graphs. These graph classes are chosen because of their hybrid structural characteristics, combining path-like, cycle-based, and attachment-driven constructions. Such structures induce nontrivial adjacency patterns that significantly influence gcd-based labeling constraints, making them suitable candidates for examining the interplay between prime and odd prime labelings.

The main contributions of this work are threefold: (i) we construct explicit prime labelings for the considered graph families, (ii) we prove the existence of odd prime labelings for these graphs, and (iii) we provide further evidence supporting the odd prime graph conjecture. Consequently, the results extend several existing studies and enlarge the class of graphs known to admit both prime and odd prime labelings.

The verification of prime and odd prime labelings is based on demonstrating that the labels assigned to adjacent vertices are relatively prime. For this purpose, we employ the following elementary number-theoretic lemma, which will be used repeatedly in subsequent proofs.

Lemma 1.1. *Let a and b be positive integers.*

a. If a and b are consecutive integers, then $\gcd(a, b) = 1$.
b. If a is odd and $c = 2^k$ for some integer $k \geq 0$, then $\gcd(a, a + c) = 1$.

Proof. See [26, 27]. □

2. Results and Discussion

In this section, we present prime and odd prime labeling constructions for the graph families considered in this study. By defining explicit vertex-labeling functions and verifying the corresponding coprimality conditions, we establish the existence of prime and odd prime labelings for comb, disjoint unions of comb, triangular book, modified triangular book graphs, and torch graphs. These results enlarge the family of graphs known to admit prime and odd prime labelings and provide further evidence in support of the odd prime graph conjecture.

2.1. *Comb and Its Disjoint Union*

The comb Cb_n consists of a path together with one pendant vertex attached to each internal path vertex. More generally, the disjoint union of m copies of the comb graph is denoted by mCb_n. The vertex set and edge set of mCb_n are defined by

$$V(mCb_n) = \{v_j^i \mid i = 1, 2, \ldots, m;\ j = 0, 1, \ldots, n\} \cup \{u_j^i \mid i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n\},$$

and

$$\begin{aligned} E(mCb_n) = &\{v_j^i v_{j+1}^i \mid i = 1, 2, \ldots, m;\ j = 0, 1, \ldots, n-1\} \\ &\cup \{u_j^i v_j^i \mid i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n\}. \end{aligned}$$

Clearly, mCb_n has order $\mid V(mCb_n) \mid = m(2n + 1)$.

Notice that the case $m = 1$ corresponds to the ordinary comb graph Cb_n.

Theorem 2.1. *For every $n \geq 2$ and $m \geq 1$, the disjoint union of m copies of the comb Cb_n, denoted by mCb_n, is a prime graph.*

Proof. Define a bijective function $f : V(mCb_n) \to \{1, 2, \ldots, 2nm + m\}$ by

$$f(v_j^i) = \begin{cases} (i-1)(2n+1) + 1, & \text{if } i = 1, 2, ..., m; j = 0, \\ (i-1)(2n+1) + 2j, & \text{if } i \equiv 0 \pmod 2,\ j = 1, 2, \ldots, n, \\ (i-1)(2n+1) + 2j + 1, & \text{if } i \equiv 1 \pmod 2,\ j = 1, 2, \ldots, n, \end{cases} \tag{1}$$

and

$$f(u_j^i) = \begin{cases} (i-1)(2n+1) + 2j + 1, & \text{if } i \equiv 0 \pmod 2,\ j = 1, 2, \ldots, n, \\ (i-1)(2n+1) + 2j, & \text{if } i \equiv 1 \pmod 2,\ j = 1, 2, \ldots, n. \end{cases} \tag{2}$$

Since every component is assigned a distinct interval of integers of length $2n+1$, no two vertices receive the same label. Hence, f is injective. Moreover, $|V(mCb_n)| = m(2n+1)$, which equals the cardinality of the codomain. Therefore, f is bijective.

Next, we verify the coprimality condition. For every edge $v_j^i v_{j+1}^i$, the assigned labels differ by either one or two according to the parity of the component index i. Consequently, one label is even and the other is odd. By Lemma 1.1,

$$\gcd\big(f(v_j^i), f(v_{j+1}^i)\big) = 1.$$

Furthermore, for every pendant edge $u_j^i v_j^i$, the labels are consecutive integers. Hence,

$$\gcd\big(f(u_j^i), f(v_j^i)\big) = 1.$$

Since there are no edges joining distinct components, every adjacent pair of vertices receives relatively prime labels. Therefore, f is a prime labeling of mCb_n. Consequently, mCb_n is a prime graph for every $n \geq 2$ and $m \geq 1$. □

Theorem 2.2. *For every $n \geq 2$ and $m \geq 1$, the disjoint union of m copies of the comb graph Cb_n, denoted by mCb_n, is an odd prime graph.*

Proof. We establish an odd prime labeling for mCb_n by defining the bijective mapping

$$f : V(mCb_n) \to \{1, 3, \dots, 4nm + 2m - 1\}$$

as follows:

$$f(v_j^i) = \begin{cases} 2(i-1)(2n+1)+1, & \text{if } i = 1, 2, ..., m; j = 0, \\ 2(i-1)(2n+1)+4j-1, & \text{if } i \equiv 0 \pmod 2,\ j = 1, 2, \dots, n, \\ 2(i-1)(2n+1)+4j+1, & \text{if } i \equiv 1 \pmod 2,\ j = 1, 2, \dots, n. \end{cases} \tag{3}$$

and

$$f(u_j^i) = \begin{cases} 2(i-1)(2n+1)+4j+1, & \text{if } i \equiv 0 \pmod 2,\ j = 1, 2, \dots, n, \\ 2(i-1)(2n+1)+4j-1, & \text{if } i \equiv 1 \pmod 2,\ j = 1, 2, \dots, n. \end{cases} \tag{4}$$

Clearly, f assigns distinct odd integers to all vertices. Since $|V(mCb_n)| = m(2n+1)$, the mapping is bijective. For every pendant edge,

$$|f(u_j^i) - f(v_j^i)| = 2.$$

Hence, by Lemma 1.1,

$$\gcd\big(f(u_j^i), f(v_j^i)\big) = 1.$$

Likewise,

$$|f(v_j^i) - f(v_{j+1}^i)| = 4,$$

for every path edge. Since 4 is a power of two, Lemma 1.1 yields

$$\gcd\big(f(v_j^i), f(v_{j+1}^i)\big) = 1.$$

Finally,

$$\gcd\big(f(v_0^i), f(v_1^i)\big) = 1$$

holds for every component by construction.

Therefore, every adjacent pair of vertices receives relatively prime labels. Hence, f is an odd prime labeling of mCb_n. Consequently, mCb_n is an odd prime graph for every $n \geq 2$ and $m \geq 1$. $\square$

2.2. Triangle Book Graphs and Their Modifications

We define the triangle book $K_{1,1,n}$ as a graph of order $n+2$ and size $2n+1$, with vertex set and edge set specified as follows:

$$V(K_{1,1,n}) = \{v_0'\} \cup \{v_i \mid i = 0, 1, \ldots, n\},$$

$$E(K_{1,1,n}) = \{v_0 v_0'\} \cup \{v_0 v_i,\ v_0' v_i \mid i = 1, 2, \ldots, n\}.$$

Theorem 2.3. *For every $n \geq 2$, the triangle book graph $K_{1,1,n}$ is a prime graph.*

Proof. Let p denote the largest prime number in the set $\{1, 2, \ldots, n+2\}$. Define a bijection $f : V(K_{1,1,n}) \to \{1, 2, \ldots, n+2\}$ by

$$f(v_0') = 1,$$

$$f(v_0) = p,$$

$$f(v_n) = 2,$$

and

$$f(\{v_1, v_2, \ldots, v_{n-1}\}) = \{3, 4, \ldots, n+2\} \setminus \{p\},$$

Clearly, the function f is bijective. Next, we verify the coprimality condition. Since $f(v_0') = 1$, we have $\gcd(f(v_0'), f(v_i)) = 1$ for every $i = 0, 1, \ldots, n$. Furthermore, $f(v_0) = p$, where p is the largest prime number in $\{1, 2, \ldots, n+2\}$. By Bertrand's Postulate, $p > \frac{n+2}{2}$. Therefore, every positive integer in $\{2, 3, \ldots, n+2\} \setminus \{p\}$ is strictly smaller than $2p$. Since p is prime, none of these integers is divisible by p. Consequently, $\gcd(p, k) = 1$ for every assigned label $k \neq p$. Hence, $\gcd(f(v_0), f(v_i)) = 1, \qquad i = 1, 2, \ldots, n$. Thus every pair of adjacent vertices receives relatively prime labels. Therefore, f is a prime labeling of $K_{1,1,n}$. Consequently, the triangle book $K_{1,1,n}$ is a prime graph for every $n \geq 2$. $\square$

Theorem 2.4. *For every $n \geq 2$, the triangle book $K_{1,1,n}$ is an odd prime graph.*

Proof. Let p denote the largest prime number in the set $\{1, 3, 5, \ldots, 2n+3\}$. Define a bijection $f : V(K_{1,1,n}) \to \{1, 3, 5, \ldots, 2n+3\}$ by

$$f(v_0') = 1,$$

$$f(v_0) = p,$$

$$f(v_n) = 3,$$

and

$$f(\{v_1, v_2, \ldots, v_{n-1}\}) = \{5, 7, \ldots, 2n+3\} \setminus \{p\},$$

Clearly, f is a bijection. Next, we verify the coprimality condition. Since $f(v_0') = 1$, we immediately obtain $\gcd(f(v_0'), f(v_i)) = 1, \quad i = 0, 1, \ldots, n$. Furthermore, $f(v_0) = p$, where p is the largest prime number in $\{1, 3, \ldots, 2n+3\}$. By Bertrand's Postulate, $p > \frac{2n+3}{2}$. Every remaining assigned label is an odd integer strictly smaller than p or an odd integer greater than p but less than $2p$. Hence, none of these labels is divisible by p. Consequently, $\gcd(p, k) = 1$ for every assigned label $k \neq p$. Therefore, $\gcd(f(v_0), f(v_i)) = 1, \quad i = 1, 2, \ldots, n$. Thus every edge of $K_{1,1,n}$ joins two vertices whose labels are relatively prime. Hence, f is an odd prime labeling of $K_{1,1,n}$. Consequently, the triangle book graph $K_{1,1,n}$ is an odd prime graph for every $n \geq 2$. □

Let a star S_m be taken in two copies, and let the central vertex of each copy be attached to the vertices v_0' and v_n, respectively. Then a new graph $K_{1,1,n} \odot 2S_m$ is obtained. An illustration of the graph $K_{1,1,n} \odot 2S_m$ is shown in Figure 1.

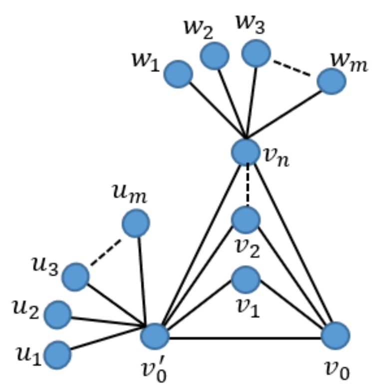

Figure 1. Illustration of the graph $K_{1,1,n} \odot 2S_m$.

Theorem 2.5. *For every $n \geq 2$ and $m \geq 1$, a graph $K_{1,1,n} \odot 2S_m$ is a prime.*

Proof. From Theorem 2.3, let $f : V(K_{1,1,n}) \to \{1, 2, \ldots, n+2\}$ be a prime labeling of the triangle book graph. Define a function $g : V(K_{1,1,n} \odot 2S_m) \to \{1, 2, \ldots, n+2m+2\}$ by

$$g(x) = f(x), \qquad \text{for every } x \in V(K_{1,1,n)}.$$

For the pendant vertices attached to v_0' and v_n, define

$$g(u_j) = \begin{cases} n+2j+2, & \text{if } n \text{ is even,} \\ n+2j+1, & \text{if } n \text{ is odd,} \end{cases} \qquad j = 1, 2, \ldots, m, \tag{5}$$

and

$$g(w_j) = \begin{cases} n+2j+1, & \text{if } n \text{ is even,} \\ n+2j+2, & \text{if } n \text{ is odd,} \end{cases} \qquad j = 1, 2, \ldots, m. \tag{6}$$

Clearly, every label in $\{1, 2, \ldots, n+2m+2\}$ is assigned exactly once. Hence, g is bijective. Since g coincides with the prime labeling f on $K_{1,1,n}$, every edge of the triangle book graph joins vertices with relatively prime labels. Next, consider the newly added edges. From Theorem 2.3, $g(v_0') = 1$ and $g(v_n) = 2$. Therefore, $\gcd(g(v_0'), g(u_j)) = \gcd(1, g(u_j)) = 1$ for every $j = 1, 2, \ldots, m$. Moreover, by construction, every label assigned to w_j is odd. Hence, $\gcd(g(v_n), g(w_j)) = \gcd(2, g(w_j)) = 1$, for every $j = 1, 2, \ldots, m$. Thus every newly introduced edge also joins two vertices with relatively prime labels. Consequently, every adjacent pair of vertices in $K_{1,1,n} \odot 2S_m$ receives relatively prime labels. Therefore, g is a prime labeling of $K_{1,1,n} \odot 2S_m$. Hence, the graph $K_{1,1,n} \odot 2S_m$ is a prime for every $n \geq 2$ and $m \geq 1$. □

Theorem 2.6. *For every $n \geq 2$ and $m \geq 1$, the graph $K_{1,1,n} \odot 2S_m$ is an odd prime.*

Proof. From Theorem 2.4, let $f : V(K_{1,1,n}) \to \{1, 3, \ldots, 2n+3\}$ be an odd prime labeling of the triangle book graph. Define $g : V(K_{1,1,n} \odot 2S_m) \to \{1, 3, \ldots, 2n+4m+3\}$ by

$$g(x) = f(x), \qquad \text{for every } x \in V(K_{1,1,n}).$$

For the vertices of the two attached star graphs, define the labels as follows. If

$$n \equiv 0 \pmod 3,$$

then

$$g(u_j) = \begin{cases} 2n+4j+3, & j \not\equiv 2 \pmod 3, \\ 2n+4j+1, & j \equiv 2 \pmod 3, \end{cases} \tag{7}$$

and

$$g(w_j) = \begin{cases} 2n+4j+1, & j \not\equiv 2 \pmod 3, \\ 2n+4j+3, & j \equiv 2 \pmod 3. \end{cases} \tag{8}$$

If

$$n \equiv 1 \pmod 3,$$

then

$$g(u_j) = \begin{cases} 2n+4j+3, & j \not\equiv 0 \pmod 3, \\ 2n+4j+1, & j \equiv 0 \pmod 3, \end{cases} \tag{9}$$

and

$$g(w_j) = \begin{cases} 2n+4j+1, & j \not\equiv 0 \pmod 3, \\ 2n+4j+3, & j \equiv 0 \pmod 3). \end{cases} \tag{10}$$

Finally, if

$$n \equiv 2 \pmod 3,$$

then

$$g(u_j) = \begin{cases} 2n+4j+3, & j \not\equiv 1 \pmod 3, \\ 2n+4j+1, & j \equiv 1 \pmod 3, \end{cases} \tag{11}$$

and

$$g(w_j) = \begin{cases} 2n+4j+1, & j \not\equiv 1 \pmod 3, \\ 2n+4j+3, & j \equiv 1 \pmod 3). \end{cases} \tag{12}$$

Since g coincides with the odd prime labeling f on $K_{1,1,n}$, every edge belonging to the triangle book graph joins vertices with relatively prime labels. Furthermore, by Theorem 2.4, $g(v_0') = 1$ and $g(v_n) = 3$. Hence, $\gcd(g(v_0'), g(u_j)) = 1$, for every $j = 1, \ldots, m$. It remains to verify the edges joining v_n and w_j. Observe that every label assigned to w_j is congruent to either 1 or 2 modulo 3. Consequently, $3 \nmid g(w_j)$, for every j. Since $g(v_n) = 3$, we obtain $\gcd(g(v_n), g(w_j)) = 1, \quad j = 1, \ldots, m$. Therefore, every newly added edge joins vertices whose labels are relatively prime. Consequently, every adjacent pair of vertices of $K_{1,1,n} \odot 2S_m$ receives relatively prime odd labels. Hence, g is an odd prime labeling of $K_{1,1,n} \odot 2S_m$. Therefore, $K_{1,1,n} \odot 2S_m$ is an odd prime graph for every $n \geq 2$ and $m \geq 1$. □

2.3. Torch Graphs

Wilson and Jini in [19] proved that the torch graph is a prime graph. Therefore, this section discusses an odd prime labeling of the torch graph. The vertex set and edge set of the torch graph are defined as follows:

$$V(O_n) = \{v_i \mid i = 1, 2, \ldots, n+4\},$$

$$\begin{aligned} E(O_n) = &\{v_i v_{n+1},\ v_i v_{n+3} \mid i = 1, 2, \ldots, n-2\} \\ &\cup \{v_1 v_i \mid i = n, n+1, \ldots, n+4\} \cup \{v_{n+3} v_{n+1},\ v_{n-1} v_n,\ v_n v_{n+2},\ v_n v_{n+4}\}. \end{aligned}$$

The structure of the torch graph is shown in Figure 2.

Theorem 2.7. *For every $n \geq 3$, the torch graph O_n is an odd prime.*

Proof. Let p denote the largest prime number in the set $\{1, 3, 5, \ldots, 2n+7\}$. Define a bijection $f : V(O_n) \to \{1, 3, 5, \ldots, 2n+7\}$ by

$$f(v_1) = 11,$$

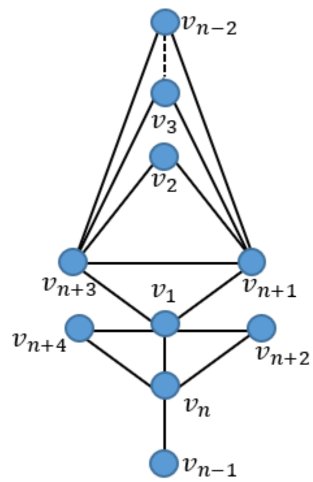

Figure 2. Structure of the torch graph O_n.

$$f(v_{n-2}) = 13,$$
$$f(v_{n-1}) = 9,$$
$$f(v_n) = 7,$$
$$f(v_{n+1}) = 1,$$
$$f(v_{n+2}) = 5,$$
$$f(v_{n+3}) = 2n + 7,$$
$$f(v_{n+4}) = 3,$$

and

$$f(\{v_2, v_3, \ldots, v_{n-3}\}) = \{13, 15, \ldots, 2n + 7\} \setminus \{p\},$$

Based on the same arguments as in the previous theorems, it follows that every pair of adjacent vertices receives relatively prime labels. Hence, f is an odd prime labeling of O_n. Therefore, the torch O_n is an odd prime graph for every $n \geq 3$. □

In the study of odd prime graph labeling, the structure of mutually non-adjacent vertex sets plays an important role in determining the independence number of a graph. Based on this observation, we obtain the following result. Before presenting the following lemma, we recall the definition of the independence number of a graph.

Definition 2.1. Let $G = (V, E)$ be a graph. An *independent set* of G is a subset $S \subseteq V(G)$ such that no two vertices in S are adjacent. The *independence number* of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of G, namely,

$$\alpha(G) = \max \{|S| : S \subseteq V(G) \text{ is an independent set}\}.$$

Lemma 2.1. *If G is an odd prime graph with $|V(G)| = n$, then the independence number satisfies $\alpha(G) \geq \lfloor \frac{n+1}{3} \rfloor$.*

Proof. Let G be an odd prime graph with $|V(G)| = n$. Then there exists a bijective labeling

$$f : V(G) \to \{1, 3, 5, \ldots, 2n - 1\}$$

such that for every edge $uv \in E(G)$, we have $\gcd(f(u), f(v)) = 1$.

Consider the subset of labels consisting of odd multiples of 3, which can be written explicitly as

$$S = \{3, 9, 15, \ldots, 6k - 3\}.$$

Take any two vertices $u, v \in V(G)$ such that $f(u), f(v) \in S$. Then there exist integers a, b such that $f(u) = 3a$ and $f(v) = 3b$. Hence,

$$\gcd(f(u), f(v)) \geq 3 > 1.$$

Since in an odd prime graph every pair of adjacent vertices must have relatively prime labels, no two vertices in S can be adjacent. Therefore, S is an independent set of G.

Next, the number of elements in S is determined by the number of odd multiples of 3 not exceeding $2n - 1$, i.e., numbers of the form $6k - 3 \leq 2n - 1$. This gives

$$k \leq \frac{n+1}{3},$$

and thus

$$|S| = \left\lfloor \frac{n+1}{3} \right\rfloor.$$

Since $S \subseteq V(G)$ is an independent set, we obtain

$$\alpha(G) \geq |S| = \left\lfloor \frac{n+1}{3} \right\rfloor.$$

Therefore, $\alpha(G) \geq \lfloor \frac{n+1}{3} \rfloor$. □

The results of this study show that the graph families Cb_n, mCb_n, $K_{1,1,n}$, $K_{1,1,n} \odot 2S_m$, and O_n admit both prime and odd prime labelings. These findings extend the class of graphs known to satisfy the Odd prime graph Conjecture proposed by Prajapati and Shah [20], which states that every prime graph is also an odd prime graph.

The labeling constructions rely on simple arithmetic patterns, including consecutive integers, parity-based assignments, and controlled shifts of integer sequences. These strategies ensure that adjacent vertices receive relatively prime labels while maintaining bijectivity. For comb graphs and their disjoint unions, the path-like structure and pendant vertices facilitate systematic label assignments. In the case of disconnected graphs mCb_n, interval-shifting techniques allow each component to preserve the required labeling properties without conflicts.

For triangular book graphs and their modifications, the existence of highly connected central vertices plays a crucial role. Assigning the label 1 or an appropriate prime number to a hub vertex simplifies the verification of coprimality conditions. Meanwhile, the odd prime labeling of $K_{1,1,n} \odot 2S_m$ demonstrates that modular arithmetic considerations can be effectively incorporated to avoid common factors among adjacent labels.

A particularly significant contribution of this work is Theorem 2.7, which establishes an odd prime labeling for torch graphs. Since torch graphs were previously known only to admit prime labelings [19], this result provides new evidence supporting the close relationship between prime and odd prime labelability.

In addition, Lemma 2.1 establishes a lower bound for the independence number of odd prime graphs, namely

$$\alpha(G) \geq \left\lfloor \frac{n+1}{3} \right\rfloor .$$

This result reveals a connection between odd prime labeling and a fundamental graph invariant, providing a new structural perspective on odd prime graphs.

Overall, all graph families investigated in this study satisfy both prime and odd prime labeling properties. These findings provide further support for the Odd prime graph Conjecture and suggest that graph structures containing path-like components, hub vertices, and pendant attachments are promising candidates for future studies on odd prime labelings.

3. Conclusion

This paper established prime and odd prime labelings for comb graphs, disjoint unions of comb graphs, triangular book graphs, modified triangular book graphs, and torch graphs through explicit labeling constructions. These results enlarge the family of graphs known to admit odd prime labelings and provide additional evidence supporting the Odd prime graph Conjecture.

In addition, a general lower bound for the independence number of odd prime graphs was obtained, revealing a connection between odd prime labeling and graph structural properties. This relationship offers a new perspective for studying odd prime graphs beyond labeling existence problems.

The findings contribute to the understanding of the interplay between graph structure and coprimality-based labelings. Future work may focus on identifying broader structural conditions that guarantee odd prime labelings and on extending these techniques to more general graph families.

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