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Partition dimension of graphs with two bridges on rose graphs

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Abstract

The partition dimension of a graph G, denoted by $pd(G)$ is a generalization of the metric dimension, in which the distinction between vertices is no longer based on a specific set of vertices, but rather on a partition of the vertex set of the graph. A partition is called a resolving partition if every vertex in the graph has a distinct distance vector representation with respect to each subset in the partition. The minimum cardinality of such a resolving partition is called the partition dimension of the graph. This study focuses on the partition dimension of double bridge graphs constructed from a pair of rose graphs. It is shown that the partition dimension of the double bridge graph obtained from two rose graphs connected by two bridge edges is 4.

Keywords: graph, partition dimension, double bridge graph, rose graph
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1. Introduction

Graph theory is a branch of mathematics that studies relationships among objects through the representation of vertices and edges [1]. This concept plays an important role in various fields of science and technology, as it can model diverse structures and relations, such as computer networks, communication systems, molecular structures, and social interactions among individuals [4]. One application of graph theory is the shortest path algorithm, which is used to determine optimal routes in a network. For example [5], a study by Koritsoglou et al. developed a penalty-based

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shortest path algorithm for pedestrian navigation systems. This algorithm modifies the k-*shortest paths* method to generate alternative routes that are not only efficient but also safer and more accessible for users. Metric dimension and partition dimension are distance-based graph parameters that can be used to uniquely identify the positions of vertices in a network. Therefore, these concepts have potential applications in navigation and network-related problems that rely on shortest-path distances. In graph theory, one of the extensively studied concepts is the metric dimension, defined as the minimum cardinality of a resolving set, and the cardinality of a metric basis [2]. This concept was later extended to the partition dimension, introduced by Chartrand et al. [3] as a variation of the metric dimension using vertex partitions.

Let G be a connected graph with vertex set $V(G)$. A partition of $V(G)$ is denoted by $\Pi = \{L_1, L_2, \ldots, L_k\}$. The partition Π is called a resolving partition if every vertex in G has a unique representation with respect to the partition. The distance from a vertex u to a subset L is denoted by $d(u, L)$, and defined as $\min\{d(u, l_i) \mid l_i \in L\}$, where l_i is a vertices in the subset L. The minimum number k of subsets in such a resolving partition is called the partition dimension of G, denoted by $pd(G)$ [3].

From the results established in [3] by Chartrand et al. [3], it follows that any graph that is not a path has partition dimension at least 3. The statement is given in the following proposition.

Proposition 1.1. *[3] Let G be a connected graph of order $n \geq 2$. Then $pd(G) = 2$ if and only if $G = P_n$.*

Furthermore, it is shown that whenever two vertices have identical distance representations, they must not lie in the same partition class; instead, they must be placed in distinct subsets.

Lemma 1.1. *[3] Let Π be a resolving partition of $V(G)$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then u and v belong to distinct elements of Π.*

A graph with two bridges is a graph formed by adding two connecting edges between two connected graphs. In this study, the focus is on a graph with two bridges denoted by $B(G, H, e_1, e_2)$. Let G and H be two connected graphs. Take edges $e_1 = u_1u_2 \in E(G)$ and $e_2 = v_1v_2 \in E(H)$. A new graph $B(G, H, e_1, e_2)$ is formed from graph G and graph H by connecting vertex u_1 to vertex v_1, producing a new edge $e_1* = u_1v_1$, and connecting vertex u_2 to vertex v_2, producing a new edge $e_2* = u_2v_2$.

A rose graph is known as the middle graph of a cycle. The middle graph $M(G)$ of a connected graph G is a graph whose vertex set is $V(G) \cup E(G)$, where two vertices are adjacent if and only if they are adjacent in G or one of them is a vertex of G and the other is an edge incident to that vertex in G. Let $v_1, v_2, \ldots, v_n$ be the vertices of a cycle C_n with $n \geq 3$, and the n edges of C_n are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$. Thus, the rose graph $M(C_n)$ can be constructed from the cycle C_n with vertices $v_1, v_2, \ldots, v_n$ by adding isolated vertices $w_1, w_2, \ldots, w_n$, and then connecting each pair of vertices v_i and v_{i+1} to w_i, for $i = 1, 2, \ldots, n$, where $v_{n+1} = v_1$. Therefore, the rose graph $M(C_n)$ has n vertices of degree 2 and n vertices of degree 4 [6].

The main objective of this study is to determine the exact value of the partition dimension of a two-bridge graph constructed from a pair of rose graphs. This research also analyzes how the presence of two connecting edges affects the ability of a partition to distinguish every vertex in the resulting graph. The results obtained are expected to enrich the study of bridge graphs and provide a deeper understanding of the characteristics of the partition dimension in graph structures composed of rose graphs.

For illustration, an example of the $M(C_n)$ of a cycle is presented in Figure 1. Figure 1 shows the graph $M(C_6)$, which is obtained from the cycle C_4 by inserting a new vertices corresponding to each edge of C_6 and connecting vertices according to the definition of a middle graph.

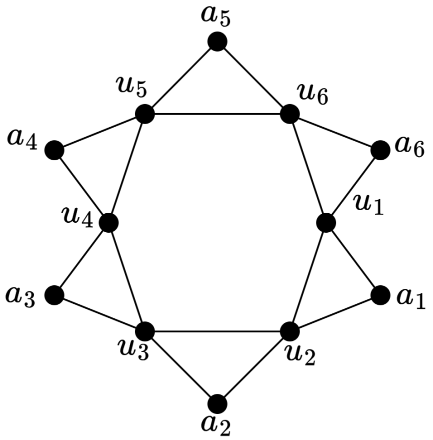

Figure 1. Rose Graph $M(C_6)$.

2. Results

This section presents the results of the study on the partition dimension of double bridge graphs constructed from a pair of rose graphs. The analysis focuses on the structural characteristics of the graph obtained after two rose graphs are connected by two distinct bridge edges. Based on this structure, the exact value of the partition dimension of the resulting graph is determined through mathematical proofs. To clarify the construction of the resolving partition employed, illustrative examples are also provided to support the obtained results.

Theorem 2.1. *Let* $M(C_m)$ *and* $M(C_n)$*be rose graphs of order* $m \geq n \geq 3$ *where* $e_1 \in E(M(C_m))$ *and* $e_2 \in E(M(C_n))$. *Then* $pd(B(M(C_m), M(C_n), e_1, e_2)) = 4$.

Proof. Let $M(C_m)$ and $M(C_n)$ be rose graphs, respectively. Their vertex sets are $V((M(C_m)) = \{u_i | 1 \leq i \leq m\} \cup \{a_i | 1 \leq i \leq m\}$ and $V((M(C_n) = \{v_i | 1 \leq i \leq n\} \cup \{b_i | 1 \leq i \leq n\}$. The corresponding edge sets are defined as above. Moreover, let $e_1 = u_1 v_1$ and $e_2 = u_2 v_2$ denote two bridge edges joining $M(C_m)$ and $M(C_n)$.

For $m \leq 5$, we define partition $\Pi = \{L_1, L_2, L_3, L_4\}$ such that

$$\begin{aligned}
L_1 &= \{u_1, u_5, a_1, a_5, v_1, v_5, b_1, b_5\},\\
L_2 &= \{u_2, u_3, a_2, v_2, v_3, b_2\},\\
L_3 &= \{u_4, a_3, a_4\},\\
L_4 &= \{v_4, b_3, b_4\}.
\end{aligned}$$

The cases $n = 3$ or $n = 4$ are handled similarly by omitting vertices that do not exist in the corresponding graph. We will show that Π is a resolving partition. Let x and y be any two distinct vertices of $V(B(M(C_m), M(C_n), e_1, e_2))$.

1. If $x, y \in L_1$, then $d(x, L_3) \neq d(y, L_3)$ or $d(x, L_4) \neq d(y, L_4)$.
2. If $x, y \in L_2$, then $d(x, L_3) \neq d(y, L_3)$ or $d(x, L_4) \neq d(y, L_4)$.
3. If $x, y \in L_3$, then $d(x, L_1) \neq d(y, L_1)$ or $d(x, L_2) \neq d(y, L_2)$.
4. If $x, y \in L_4$, then $d(x, L_1) \neq d(y, L_1)$ or $d(x, L_2) \neq d(y, L_2)$.

Thus, Π is a resolving partition, and we conclude that $pd(B(M(C_m), M(C_n), e_1, e_2)) \leq 4$.

For $m \geq 6$, we define partition $\Pi = \{L_1, L_2, L_3, L_4\}$ such that

$$\begin{aligned}
L_1 &= \{u_i, u_m, a_1, a_j, v_k, v_n, b_1, b_l\} \text{ for } 1 \leq i \leq \lceil \tfrac{m-2}{4} \rceil,\ m-1 \leq j \leq m,\ 1 \leq k \leq \lceil \tfrac{n-2}{4} \rceil,\\
&\quad n-1 \leq l \leq n,\\
L_2 &= \{u_i, a_j, v_k, b_l\} \text{ for } \lceil \tfrac{m-2}{4} \rceil + 1 \leq i \leq \lceil \tfrac{m}{2} \rceil,\ 2 \leq j \leq \lceil \tfrac{m}{2} \rceil,\ \lceil \tfrac{n-2}{4} \rceil + 1 \leq k \leq \lceil \tfrac{n}{2} \rceil,\\
&\quad 2 \leq l \leq \lceil \tfrac{n}{2} \rceil,\\
L_3 &= \{u_i, a_j\} \text{ for } \lceil \tfrac{m}{2} \rceil + 1 \leq i \leq m-1,\ \lceil \tfrac{m}{2} \rceil + 1 \leq j \leq m-2,\\
L_4 &= \{v_i, b_j\} \text{ for } \lceil \tfrac{n}{2} \rceil + 1 \leq i \leq n-1,\ \lceil \tfrac{n}{2} \rceil + 1 \leq j \leq n-2.
\end{aligned}$$

The above partition construction is valid for all $m \geq n \geq 3$, including the case $m > n$, since the vertices of $M(C_m)$ and $M(C_n)$ are partitioned independently according to their respective orders. We will show that Π is a resolving partition. Let x and y be any two distinct vertices of $V(B(M(C_m), M(C_n), e_1, e_2))$.

1. If $x, y \in L_1$, then $d(x, L_3) \neq d(y, L_3)$ or $d(x, L_4) \neq d(y, L_4)$.
2. If $x, y \in L_2$, then $d(x, L_3) \neq d(y, L_3)$ or $d(x, L_4) \neq d(y, L_4)$.
3. If $x, y \in L_3$, then $d(x, L_1) \neq d(y, L_1)$ or $d(x, L_2) \neq d(y, L_2)$.
4. If $x, y \in L_4$, then $d(x, L_1) \neq d(y, L_1)$ or $d(x, L_2) \neq d(y, L_2)$.

Thus, Π is a resolving partition, and we conclude that $pd(B(M(C_m), M(C_n), e_1, e_2)) \leq 4$.

We first establish a lower bound for $pd(B(M(C_m), M(C_n), e_1, e_2))$, since $pd(M(C_m)) \geq pd(M(C_n))$ is not a path, by Proposition 1.1 then $pd(B(M(C_m), M(C_n), e_1, e_2)) \geq 3$. This means that there exists $\Pi = L_1, L_2, L_3$ such that for every two vertices $x, y \in V(B(M(C_m), M(C_n), e_1, e_2))$, it holds that $r(x|\Pi) \neq r(y|\Pi)$. Since there are only three partition classes, there are eight possible ways to partition the vertices u_1, u_2, v_1, v_2, and several cases to consider for the partitions of a_1 and b_1, namely as follows

- Case 1: $u_1 \in L_1, u_2 \in L_1, v_1 \in L_1, v_2 \in L_1$. In this case, there are four possible configurations for the placements of a_1 and b_1, namely as follows

  (i) $a_1 \in L_1, b_1 \in L_1$,
  (ii) $a_1 \in L_2, b_1 \in L_2$,
  (iii) $a_1 \in L_1, b_1 \in L_2$,
  (iv) $a_1 \in L_2, b_1 \in L_3$.

  For case (i), assume that $d(u_1, L_2) = d(u_2, L_2)$. Then $d(u_1, L_3) \neq d(u_2, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(u_2, L_3)$. Consequently, one of the following conditions occurs
  $$r(u_2|\Pi) = r(v_1|\Pi),$$
  $$r(a_1|\Pi) = r(v_1|\Pi),$$
  $$r(a_1|\Pi) = r(u_2|\Pi).$$

  This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (ii), observe that the four vertices u_1, u_2, v_1, and v_2 have the same distance to L_2. Therefore, their distances to L_3 must all be distinct. Without loss of generality, suppose that $d(u_1, L_3) < d(u_2, L_3) < d(v_1, L_3) < d(v_2, L_3)$. Consequently, $r(u_2|\Pi) = r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (iii), observe that the vertices u_1 and u_2 have the same distance to L_2. Hence, $d(u_1, L_3) \neq d(u_2, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(u_2, L_3)$. Consequently, one of the following conditions occurs

  $$r(u_2|\Pi) = r(v_1|\Pi),$$
  $$r(u_2|\Pi) = r(a_1|\Pi),$$
  $$r(u_2|\Pi) = r(a_m|\Pi).$$

  This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (iv), observe that the vertices u_1 and u_2 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(u_2, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(u_2, L_3)$. Similarly, since v_1 and v_2 have the same distance to L_3, it follows that $d(v_1, L_2) \neq d(v_2, L_2)$. Consequently, $r(u_1|\Pi) = r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

- Case 2: $u_1 \in L_1, u_2 \in L_1, v_1 \in L_2, v_2 \in L_2$. In this case, there are five possible configurations for the placements of a_1 and b_1, namely as follows:

  (i) $a_1 \in L_1, b_1 \in L_1$,
  (ii) $a_1 \in L_1, b_1 \in L_2$,
  (iii) $a_1 \in L_1, b_1 \in L_3$,
  (iv) $a_1 \in L_2, b_1 \in L_3$,
  (v) $a_1 \in L_3, b_1 \in L_3$.

  For case (i), observe that the vertices u_1 and u_2 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(u_2, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(u_2, L_3)$. Consequently, $r(u_2|\Pi) = r(b_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (ii), observe that the vertices u_1 and u_2 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(u_2, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(u_2, L_3)$. Consequently, $r(v_1|\Pi) = r(v_2|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (iii), observe that the vertices v_1 and v_2 have the same distance to L_2 and L_3. Consequently, $r(v_1|\Pi) = r(v_2|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (iv), observe that the vertices u_1 and u_2 have the same distance to L_2 and L_3. Furthermore, observe that the vertices v_1 and v_2 have the same distance to L_1 and L_3. Consequently, at least one of the following equalities holds: $r(u_1|\Pi) = r(u_2|\Pi)$ or $r(v_1|\Pi) = r(v_2|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (v), observe that the vertices u_1 and u_2 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(u_2, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(u_2, L_3)$. Furthermore, observe that the vertices v_1 and v_2 have the same distance to L_1 and L_3. Consequently, $r(v_1|\Pi) = r(v_2|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

- Case 3: $u_1 \in L_1, u_2 \in L_2, v_1 \in L_1, v_2 \in L_2$. In this case, there are six possible configurations for the placements of a_1 and b_1, namely as follow

  (i) $a_1 \in L_1, b_1 \in L_1$,

  (ii) $a_1 \in L_1, b_1 \in L_2$,

  (iii) $a_1 \in L_1, b_1 \in L_3$,

  (iv) $a_1 \in L_2, b_1 \in L_2$,

  (v) $a_1 \in L_2, b_1 \in L_3$,

  (vi) $a_1 \in L_3, b_1 \in L_3$.

  For case (i), observe that the vertices u_1 and v_1 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(v_1, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(v_1, L_3)$. Consequently, $r(a_1|\Pi) = r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (ii), observe that the vertices u_1 and v_1 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(v_1, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(v_1, L_3)$. Consequently, $r(a_1|\Pi) = r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (iii), observe that the vertices u_1 and a_1 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(a_1, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(a_1, L_3)$. Consequently, $r(u_1|\Pi) = r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (iv), observe that the vertices u_1 and v_1 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(v_1, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(v_1, L_3)$. Consequently, one of the following conditions occurs

  $$r(a_1|\Pi) = r(u_2|\Pi),$$
  $$r(b_1|\Pi) = r(v_2|\Pi),$$
  $$r(u_2|\Pi) = r(v_2|\Pi).$$

  This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (v), observe that the vertices a_1 and u_2 have the same distance to L_1. Hence, it must be that $d(a_1, L_3) \neq d(u_2, L_3)$. Without loss of generality, suppose that $d(u_2, L_3) < d(a_1, L_3)$. Consequently, one of the following conditions occurs $r(u_2|\Pi) = r(v_2|\Pi)$ and

$r(u_1|\Pi) = r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (vi), observe that the vertices u_1 and v_1 have the same distance to L_1 and L_3, the vertices u_2 and v_2 have the same distance to L_1 and L_2, and the vertices a_1 and b_1 have the same distance to L_1 and L_2. Consequently, one of the following conditions occurs

$$\begin{aligned} r(u_1|\Pi) &= r(v_1|\Pi), \\ r(u_2|\Pi) &= r(v_2|\Pi), \\ r(a_1|\Pi) &= r(b_1|\Pi). \end{aligned}$$

This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

- Case 4: $u_1 \in L_1, u_2 \in L_1, v_1 \in L_1, v_2 \in L_2$. In this case, there are five possible configurations for the placements of a_1 and b_1, namely as follows

  (i) $a_1 \in L_1, b_1 \in L_1$,
  (ii) $a_1 \in L_1, b_1 \in L_2$,
  (iii) $a_1 \in L_1, b_1 \in L_3$,
  (iv) $a_1 \in L_2, b_1 \in L_2$,
  (v) $a_1 \in L_2, b_1 \in L_3$.

  For case (i), observe that the vertices u_2, v_1, and b_1 have the same distance to L_2. Hence, it must be that $d(u_2, L_3) \neq d(v_1, L_3) \neq d(b_1, L_3)$. Without loss of generality, suppose that $d(v_1, L_3) < d(b_1, L_3) < d(u_2, L_3)$. Consequently, one of the following conditions occurs

$$\begin{aligned} r(u_1|\Pi) &= r(b_1|\Pi), \\ r(u_1|\Pi) &= r(u_2|\Pi), \\ r(u_1|\Pi) &= r(v_1|\Pi). \end{aligned}$$

  This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (ii), observe that the vertices u_1 and a_1 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(a_1, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(a_1, L_3)$. Consequently, $r(b_1|\Pi) = r(v_2|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (iii), observe that the vertices u_2 and v_1 have the same distance to L_2. Hence, it must be that $d(u_2, L_3) \neq d(v_1, L_3)$. Without loss of generality, suppose that $d(v_1, L_3) < d(u_2, L_3)$. Consequently, one of the following conditions occurs $r(u_1|\Pi) = r(u_2|\Pi)$ and $r(u_1|\Pi) =$

$r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (iv), observe that the vertices u_1, u_2, and v_1 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(u_2, L_3) \neq d(v_1, L_3)$. Without loss of generality, suppose that $d(u_2, L_3) < d(u_1, L_3) < d(v_1, L_3)$. Consequently, $r(a_1|\Pi) = r(b_2|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (v), observe that the vertices u_1, u_2, and v_1 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(u_2, L_3) \neq d(v_1, L_3)$. Without loss of generality, suppose that $d(u_2, L_3) < d(u_1, L_3) < d(v_1, L_3)$. Consequently, $r(a_1|\Pi) = r(b_2|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

- Case 5: $u_1 \in L_1, u_2 \in L_2, v_1 \in L_2, v_2 \in L_1$. In this case, there are five possible configurations for the placements of a_1 and b_1, namely as follows:

  (i) $a_1 \in L_1, b_1 \in L_1$,
  (ii) $a_1 \in L_1, b_1 \in L_3$,
  (iii) $a_1 \in L_2, b_1 \in L_2$,
  (iv) $a_1 \in L_2, b_1 \in L_3$,
  (v) $a_1 \in L_3, b_1 \in L_3$.

  For case (i), observe that the vertices u_1 and a_1 have the same distance to L_2. Hence, it must be that $d(u_1, L_3) \neq d(a_1, L_3)$. Without loss of generality, suppose that $d(u_1, L_3) < d(a_1, L_3)$. Consequently, $r(u_2|\Pi) = r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (ii), observe that the vertices u_1 and a_1 have the same distance to L_2. Therefore, it must hold that $d(u_1, L_3) \neq d(a_1, L_3)$. Without loss of generality, assume that $d(u_1, L_3) < d(a_1, L_3)$. Consequently, $r(u_1|\Pi) = r(v_2|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

  For case (iii), observe that the vertices u_1 and v_2 have the same distance to L_2. Hence, it must hold that $d(u_1, L_3) \neq d(v_2, L_3)$. Without loss of generality, assume that $d(u_1, L_3) < d(v_2, L_3)$. Consequently, one of the following conditions occurs

$$r(u_2|\Pi) = r(a_1|\Pi),$$
$$r(u_2|\Pi) = r(v_1|\Pi),$$
$$r(a_1|\Pi) = r(v_1|\Pi).$$

This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (iv), observe that the vertices u_1 and v_2 have the same distance to L_2. Therefore, it must hold that $d(u_1, L_3) \neq d(v_2, L_3)$. Without loss of generality, assume that $d(v_2, L_3) < d(u_1, L_3)$. Consequently, $r(u_2|\Pi) = r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (v), observe that the vertices u_1 and v_2 have the same distance to L_2 and L_3. Moreover, the vertices u_2 and v_1 have the same distance to L_1 and L_3. Consequently, one of the following conditions occurs

$$\begin{aligned} r(u_1|\Pi) &= r(v_2|\Pi), \\ r(u_2|\Pi) &= r(v_1|\Pi), \\ r(a_1|\Pi) &= r(b_1|\Pi). \end{aligned}$$

This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

- Case 6: $u_1 \in L_1, u_2 \in L_2, v_1 \in L_2, v_2 \in L_3$. In this case, there are one possible configurations for the placements of a_1 and b_1 namely as follows

  (i) $a_1 \in L_1, b_1 \in L_1$.

  For case (i), observe that the vertices u_2 and v_1 have the same distance to L_1 and L_3. Consequently, $r(u_2|\Pi) = r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

- Case 7: $u_1 \in L_1, u_2 \in L_1, v_1 \in L_2, v_2 \in L_3$. In this case, there are four possible configurations for the placements of a_1 and b_1 namely as follows

  (i) $a_1 \in L_1, b_1 \in L_1$,
  (ii) $a_1 \in L_1, b_1 \in L_2$,
  (iii) $a_1 \in L_2, b_1 \in L_2$,
  (iv) $a_1 \in L_2, b_1 \in L_3$.

  For case (i), one of the following conditions occurs

$$\begin{aligned} r(u_1|\Pi) &= r(b_1|\Pi), \quad & r(b_1|\Pi) &= r(b_2|\Pi), \\ r(b_1|\Pi) &= r(b_n|\Pi), \quad & r(b_1|\Pi) &= r(v_3|\Pi), \\ r(b_1|\Pi) &= r(v_n|\Pi), \quad & r(b_1|\Pi) &= r(u_2|\Pi), \end{aligned}$$

  This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (ii), one of the following conditions occurs

$$\begin{array}{ll} r(u_2|\Pi) = r(b_2|\Pi) = r(a_m|\Pi), & r(u_2|\Pi) = r(b_2|\Pi) = r(u_m|\Pi), \\ r(u_2|\Pi) = r(v_3|\Pi) = r(a_m|\Pi), & r(u_2|\Pi) = r(v_3|\Pi) = r(u_m|\Pi), \\ r(u_2|\Pi) = r(v_3|\Pi) = r(v_n|\Pi), & r(u_2|\Pi) = r(b_2|\Pi) = r(b_n|\Pi), \\ r(u_2|\Pi) = r(v_3|\Pi) = r(b_n|\Pi), & r(u_2|\Pi) = r(b_2|\Pi) = r(b_n|\Pi). \end{array}$$

This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (iii), one of the following conditions occurs

$$\begin{array}{ll} r(u_2|\Pi) = r(b_2|\Pi), & r(u_1|\Pi) = r(u_2|\Pi) = r(v_3|\Pi), \\ r(u_1|\Pi) = r(u_2|\Pi) = r(b_2|\Pi), & r(u_2|\Pi) = r(b_n|\Pi) = r(b_2|\Pi), \\ r(u_2|\Pi) = r(v_n|\Pi) = r(v_3|\Pi), & r(u_2|\Pi) = r(b_n|\Pi) = r(v_3|\Pi), \\ r(u_2|\Pi) = r(v_n|\Pi) = r(b_2|\Pi). & \end{array}$$

This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (iv), one of the following conditions occurs

$$\begin{array}{ll} r(u_1|\Pi) = r(u_2|\Pi), & r(u_1|\Pi) = r(b_n|\Pi), \\ r(u_1|\Pi) = r(v_n|\Pi), & r(b_1|\Pi) = r(v_n|\Pi), \\ r(b_1|\Pi) = r(b_n|\Pi), & r(u_2|\Pi) = r(b_2|\Pi). \\ r(u_2|\Pi) = r(v_3|\Pi), & \end{array}$$

This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

- Case 8: $u_1 \in L_1, u_2 \in L_3, v_1 \in L_2, v_2 \in L_3$. In this case, there are five possible configurations for the placements of a_1 and b_1 namely as follows

  (i) $a_1 \in L_1$, $b_1 \in L_1$,

  (ii) $a_1 \in L_1$, $b_1 \in L_2$,

  (iii) $a_1 \in L_1$, $b_1 \in L_3$,

  (iv) $a_1 \in L_2$, $b_1 \in L_2$,

  (v) $a_1 \in L_3$, $b_1 \in L_3$.

For case (i), observe that the vertices u_1 and b_1 have the same distance to L_2 and L_3. Consequently, $r(u_1|\Pi) = r(b_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (ii), observe that the vertices u_1 and a_1 have the same distance to L_3. Hence, it must hold that $d(u_1, L_2) \neq d(a_1, L_2)$. Without loss of generality, suppose that $d(u_1, L_2) < d(a_1, L_2)$. Consequently, one of the following conditions occurs

$$\begin{aligned} r(u_1|\Pi) &= r(u_3|\Pi) = r(v_3|\Pi), \\ r(u_1|\Pi) &= r(u_3|\Pi) = r(b_2|\Pi), \\ r(u_1|\Pi) &= r(a_2|\Pi) = r(v_3|\Pi), \\ r(u_1|\Pi) &= r(a_2|\Pi) = r(b_2|\Pi), \\ r(u_1|\Pi) &= r(u_m|\Pi) = r(b_n|\Pi). \end{aligned}$$

This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (iii), observe that the vertices b_1 and v_2 have the same distance to L_2. Hence, it must hold that $d(b_1, L_1) \neq d(v_2, L_1)$. Without loss of generality, suppose that $d(v_2, L_1) < d(b_1, L_1)$. Consequently, one of the following conditions occurs

$$\begin{aligned} r(u_1|\Pi) &= r(v_3|\Pi) = r(b_2|\Pi), \\ r(v_1|\Pi) &= r(b_2|\Pi) = r(v_3|\Pi), \\ r(v_3|\Pi) &= r(a_1|\Pi) = r(b_2|\Pi), \\ r(u_2|\Pi) &= r(b_2|\Pi) = r(v_3|\Pi). \end{aligned}$$

This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (iv), observe that the vertices a_1 and v_1 have the same distance to L_1 and L_2. Consequently, $r(a_1|\Pi) = r(v_1|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

For case (v), observe that the vertices a_1 and u_2 have the same distance to L_1. Hence, it must hold that $d(a_1, L_2) \neq d(u_2, L_2)$. Without loss of generality, suppose that $d(u_2, L_2) < d(a_1, L_2)$. Consequently, one of the following conditions occurs $r(v_2|\Pi) = r(b_1|\Pi)$ and $r(v_2|\Pi) = r(u_2|\Pi)$. This yields a contradiction to the assumption that $pd(B(M(C_m), M(C_n), e_1, e_2)) = 3$. Hence, it follows that $pd(B(M(C_m), M(C_n), e_1, e_2)) > 3$.

From all the possibilities above, it is proven that $pd(B(M(C_6), M(C_6), e_1, e_2)) > 3$. Since $pd(B(M(C_m), M(C_n), e_1, e_2)) \geq 4$ and $pd(B(M(C_m), M(C_n), e_1, e_2)) \leq 4$, then $pd(B(M(C_m), M(C_n), e_1, e_2)) = 4$. □

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