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Tripotent graph of finite rings

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Abstract

In this paper, we find the number of non-trivial tripotent elements for some finite rings, namely $\mathbb{Z}_n$, $H(\mathbb{F}_q)$, and $\mathbb{F}_q C_n$. For this purpose, we find the general formula of them. Furthermore, we introduce the tri-potent graph of a finite ring R, denoted by $Tri(R)$, where two distinct vertices x and y in R with $a < b$ are adjacent if and only if $a - b \in Tri(R)$. It is shown that the tri-potent graph is a bi-regular connected with girth at most 4. Also, the tri potent graph of $\mathbb{Z}_n$ is bipartite graph for some n.

Keywords: Tripotent Graph, Chinese Remainder Theorem, Girth, Connectedness
Mathematics Subject Classification : 20B40

1. Introduction

The interplay between finite commutative rings and graph theory has been extensively studied in [9, 10, 12]. Anderson and Badawi [8] initiated this exploration by introducing the total graph, in which the vertices represent elements of a ring R and adjacency occurs precisely when the sum of two elements is a zero-divisor. This was followed by Ashrafi et al. [10], who introduced the unit graph, where adjacency is defined whenever the sum of two distinct vertices is a unit. Razzaghi and Sahebi [11], who introduced the idempotent graph, denoted by $G_{Id}(R)$, in which two distinct elements are adjacent if and only if their sum is idempotent. Motivated by these developments, we introduce in this paper the tri-potent graph of a finite ring R, denoted by $Tri(R)$, and examine its fundamental graphical properties, including connectedness and girth. A graph is connected if there exists a path connecting any two distinct vertices. The distance between vertices x and y is the length of the shortest path connecting them, and the diameter of the graph, $diam(\Gamma)$, is the

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supremum of these distances. The girth of a graph, denoted $g(\Gamma)$ is the length of the shortest cycle in Γ. A graph without cycles has an infinite girth. An bi-partite graph is one whose vertex set can be partitioned into two subsets so that an edge has both ends in no subset.

Throughout this paper, a ring R will denote a non-trivial finite ring with identity 1. Let G be a non-trivial finite group and RG denote the group ring of G over the ring R. Thus RG is the set of all formal linear combinations of the form $\alpha = \sum_{g\in G} a_g g$, where $a_g \in R$ and $a_g = 0$ almost everywhere, that is, only a finite number of coefficients are different from 0 in each of these sums. We define the sum and product of two elements in RG by:

$$\alpha + \beta = \sum_{g\in G}(a_g + b_g)g$$

and

$$\alpha\beta = \sum_{g,h\in G}(a_g b_h)gh.$$

If $R = K$ is a field, then KG is the group algebra of G over K. Let $C_n = \langle x \mid x^n = 1\rangle$ denote the cyclic group of order n. Quaternions, denoted by $H(R)$, were first discovered by William R. Hamilton in 1843. A quaternion is of the form $x = a_0 + a_1 i + a_2 j + a_3 k$, where $i^2 = j^2 = k^2 = ijk = -1$. Algebraically speaking, $H(R)$ forms a division algebra (skew field) over R of dimension 4. In particular, they found the number of zero divisors of a quaternion ring over $\mathbb{Z}_p$ and provided a detailed structural description of the zero divisor graph of a quaternion ring over $\mathbb{Z}_p$. The number of zero divisors and the number of idempotents of the quaternion ring can also be found in [**?**]. Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements, where p is a prime. The group ring $\mathbb{F}_q C_n$ is isomorphic to $\mathbb{F}_q[x]/\langle x^n - 1\rangle$, the quotient ring of polynomials over finite fields $\mathbb{F}_q$. In,[4], they show that the number of idempotent elements in quaternion rings on $\mathbb{Z}_p$ is $p^2 + p + 1$ for p is odd prime. In [5] Mohammed Salih gave some formula to compute the probability that the multiplication of two randomly chosen elements of a group ring RG is zero. In [3], Mosi, define k-potent element in a ring R. In particular, an element x of a ring R is called tripotent element if $x^3 = x$. A tripotent element x is said to be non trivial if $x^2 \neq x$. The number of non-trivial tripotent elements of R is denoted by $NT(R)$. The following are used to prove our results.

Theorem 1.1 (Chinese Remainder Theorem)**.** *If $m_1, \ldots, m_k$ are pairwise relatively prime positive integers, and if $a_1, \ldots, a_k$ are any integers then the simultaneous congruences $x \equiv a_1 \mod m_1$, $x \equiv a_2 \mod m_2, \ldots$, $x \equiv a_k \mod m_k$, have a solution and the solution is unique modulo m, where $m = m_1 \ldots m_k$.*

Theorem 1.2 (Hensel's Lifting Lemma)**.** *Let $f(x)$ be a polynomial with integer coefficients, p a prime, and suppose a satisfies*

$$f(a) \equiv 0 \pmod{p^j}$$

with $f'(a) \not\equiv 0 \pmod p$. Then there exists a unique integer $t \pmod p$ such that $a + tp^j$ is a solution to congruence $f(x) \equiv 0 \mod p^{j+1}$.

The paper is organized as follows: in Section 2 we determine the number of non-trivial tripotent elements of the ring of integers module n, the group algebra $\mathbb{F}_q C_n$ and $H(\mathbb{F}_q)$. In particular, we drive the general formula for computing the number of non-trivial tripotent elements of these rings. In section 3, we define the tri-potent graph on R and several graph properties are given.

2. Number of Tripotent Elements of R

In this section, we compute the number of non trivial tripotent elements in $\mathbb{Z}_n$, $\mathbb{F}_q C_n$ and $H(\mathbb{F}_q)$.

Theorem 2.1. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, then $\mathbb{Z}_n$ has*

$$NT(\mathbb{Z}_n) = \begin{cases} 3^m - 2^m, & 2 \nmid n, \\ 2 \cdot 3^{m-1} - 2^m, & p_1 = 2,\ \alpha_1 = 1, \\ 3^m - 2^m, & p_1 = 2,\ \alpha_1 = 2, \\ 5 \cdot 3^{m-1} - 2^m, & 2^3 \mid n. \end{cases}$$

non trivial tripotent elements.

Proof. Let

$$NT_2(n) = |\{x \in \mathbb{Z}_n : x^2 = x\}|, \qquad NT_3(n) = |\{x \in \mathbb{Z}_n : x^3 = x\}|.$$

and

$$NT(n) = NT_3(n) - NT_2(n).$$

We want to establish a formula for the above three questions. We first note that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, then for $j = 2, 3$.

$$NT_j(n) = \prod_{i=1}^{m} NT_j(p_i^{\alpha_i}), \tag{1}$$

This is a direct consequence of the Chinese Remainder Theorem. So we only need to compute a formula for $NT_j(p^\alpha)$. For a prime p, we have

$$NT_j(p) = \begin{cases} j, & p > 2, \\ 2, & p = 2. \end{cases}$$

Thus

$$NT(p) = \begin{cases} 1, & p > 2, \\ 0, & p = 2. \end{cases}$$

In fact $r = 0, 1, -1$ are the roots of $x^3 - x \equiv 0 \pmod{p}$ (note that if $p = 2, 1 = -1$) and $r = 0, 1$ are the roots of $x^2 - x \equiv 0 \pmod{p}$. To lift solutions modulo higher powers of p, we apply Hensel's Lemma to

$$f_3(x) = x^3 - x, \qquad f_2(x) = x^2 - x.$$

Their derivatives are

$$f_3'(x) = 3x^2 - 1, \qquad f_2'(x) = 2x - 1.$$

We obtain

$$f_3'(r) \equiv \begin{cases} -1 \pmod p, & r = 0, \\ 2 \pmod p, & r = \pm 1, \end{cases} \qquad f_2'(r) \equiv \begin{cases} -1 \pmod p, & r = 0, \\ 1 \pmod p, & r = 1. \end{cases}$$

Hence $f_j'(r) \not\equiv 0 \pmod p$ for all r, j and p except $r = \pm 1, p = 2$ and $j = 3$. We conclude that:

1. $NT_j(p^\alpha) = j$ for all $p > 3$ and for all $\alpha > 0$. Hence applying (1), if n is odd, then $NT(n) = 3^m - 2^m$.
2. $NT_2(2^\alpha) = 2$ for all $\alpha \geq 0$.

The behavior of $NT_3(2^\alpha)$ must be handled separately. One checks that:

$$NT_3(2) = 2, \quad NT_3(4) = 3, \quad NT_3(8) = 5,$$

and for $\alpha \geq 3$, exactly four roots lift at each step. Thus the stated cases follow. Substituting into Equation (1) yields the desired formula. □

Example 2.1. *In $\mathbb{Z}_{45}$, there are 5 non-trivial tripotent elements:*

$$9,\ 19,\ 26,\ 35,\ 44.$$

Theorem 2.2. *The number of non-trivial tripotent elements in $H(\mathbb{F}_q)$ is*

$$NT(H(\mathbb{F}_q)) = \begin{cases} q^3 - 1, & \text{char}(\mathbb{F}_q) = 2, \\ 2q^2 + 2q + 1, & \textit{otherwise}. \end{cases}$$

Proof. The proof is similar as Theorem 2.2 in [4]. □

Theorem 2.3. *The number of non-trivial tripotent elements in $\mathbb{F}_q C_2$ is*

$$NT(\mathbb{F}_q C_2) = \begin{cases} 5, & \gcd(q, 2) = 1, \\ q - 1, & \textit{otherwise}. \end{cases}$$

Proof. If $\gcd(q, 2) = 1$, then $\mathbb{F}_q C_2 \cong \mathbb{F}_q \oplus \mathbb{F}_q$. Since $\mathbb{F}_q$ has exactly one non-trivial tripotent element $q - 1$, it follows that the elements

$$(q-1, 0),\ (0, q-1),\ (q-1, q-1),\ (1, q-1),\ (q-1, 1)$$

are non-trivial tripotent elements, giving 5 in total. If $\gcd(q, 2) \neq 1$, then any element $y = a \cdot 1 + b \cdot x$ is tripotent if $a + b = 1$ and $a \neq b \in \mathbb{F}_q C_2 \setminus \{0, 1\}$. This gives $q - 2$ solutions. On the other hand, $(1.x)^3 = x^3 = x^2 x = x$ is non trivial tripotent element. Together with x, we obtain, $q - 1$ non-trivial tripotent elements. □

Theorem 2.4. 1. *If* $\mathrm{char}(\mathbb{F}_q) \neq 2$*, then*

$$NT(\mathbb{F}_q C_3) = \begin{cases} 1, & q \equiv 0 \pmod 3, \\ 19, & q \equiv 1 \pmod 3, \\ 5, & q \equiv 2 \pmod 3. \end{cases}$$

$$NT(\mathbb{F}_q C_5) = \begin{cases} 5, & q \equiv 2, 3 \pmod 5, \\ 19, & q \equiv 4 \pmod 5, \\ 1, & q \equiv 0 \pmod 5, \\ 211, & q \equiv 1 \pmod 5. \end{cases}$$

Otherwise, $NT(\mathbb{F}_q C_3) = NT(\mathbb{F}_q C_5) = 0$*.*

2.

$$NT(\mathbb{F}_q C_4) = \begin{cases} 19, & q \equiv 3 \pmod 4, \\ 65, & q \equiv 1 \pmod 4, \\ q^2 - 1, & q \equiv 0 \pmod 4. \end{cases}$$

3. Tripotent graph

Definition 3.1. *Let* R *be a ring. The* tripotent graph *of* R*, denoted by* $Tri(R)$*, is the simple undirected graph whose vertex set is* R*, and two distinct vertices* a *and* b *are adjacent if* $a - b \in Tri(R)$ *and* $a < b$*.*

Lemma 3.1. *The following hold:*

1. $n - 1$ *is tri-potent element in* $\mathbb{Z}_n$*.*
2. a_i *and* $a_i + 1$ *are adjacent in* $Tri(\mathbb{Z}_n)$*, where* $0 \leq a_i \leq n$*.*
3. 0 *and* $n - 1$ *are not adjacent in* $Tri(\mathbb{Z}_n)$*.*
4. *If* k *and* $k + 1$ *are tripotent and* $a - p = k$ *then* $a \sim p \sim p + 1$ *is the cycle* C_3 *in* $Tri(\mathbb{Z}_n)$*.*

Proof. Clear. □

Remark 3.1. 1. $NT(\mathbb{Z}_2)) = 0$.

2. *The tripotent graph of* $\mathbb{F}_q C_n$ *is disconnected for all, but finitely many of them.*

Example 3.1. *The tri-potent graph of* $\mathbb{Z}_4$ *is the path graph* P_4 *and*

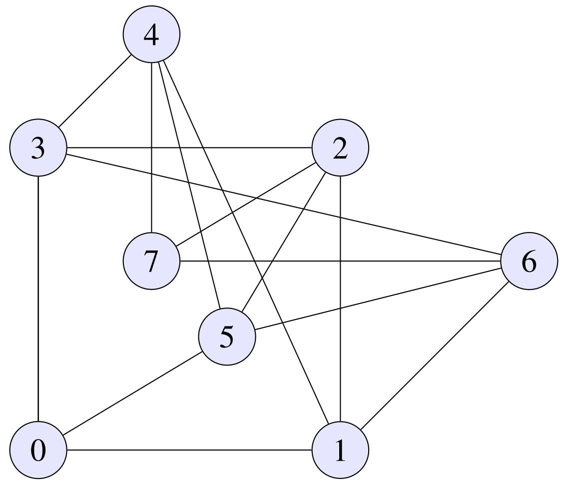

Figure 1. The tripotent graph of $\mathbb{Z}_8$

Lemma 3.2. *Let $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be the prime power decomposition of n, and let $Tri(\mathbb{Z}_n)$ be the tripotent graph of $\mathbb{Z}_n$. Denote*

$$NT(n) = |Tri(\mathbb{Z}_n)|$$

as the number of non-trivial tripotents in $\mathbb{Z}_n$. Then for any vertex $v \in V(Tri(\mathbb{Z}_n))$,

$$\deg(v) = \begin{cases} NT(n), & \text{for exactly } 2^m \text{ vertices}, \\ NT(n) + 1, & \text{for the remaining vertices}. \end{cases}$$

Proof. In the simple undirected graph $Tri(\mathbb{Z}_n)$, the neighbors of a vertex v are precisely the elements $b \in \mathbb{Z}_n$ such that $v - b \in Tri(\mathbb{Z}_n)$. By the Chinese Remainder Theorem,

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{a_1}} \times \cdots \times \mathbb{Z}_{p_m^{a_m}},$$

and non-trivial tripotents correspond to tuples of non-trivial tripotents in the components. Most vertices have no overlap with trivial tripotents and thus have degree $NT(n)$, while vertices whose component tuples include certain trivial tripotents gain an extra adjacency, giving degree $NT(n) + 1$. There are exactly $n - 2^m$ such exceptional vertices, and the remaining 2^m vertices have degree $NT(n)$. This establishes the stated degree formula for all vertices. □

Proposition 3.1. *The tri-potent graph of $\mathbb{Z}_n$ is a bi-regular connected graph.*

Proof. The proof of bi-regularity graph follows from Lemma 3.2. Let $a, b \in \mathbb{Z}_n$ be arbitrary vertices. If $a - b$ is a nontrivial tripotent in $\mathbb{Z}_n$, then a and b are adjacent, and hence there exists a path from a to b.

Suppose that $a - b$ is not a nontrivial tripotent. The set T of all nontrivial tripotents is finite. Hence, there exist elements

$$a_i, a_{i+1} \in \mathbb{Z}_n$$

such that

$$a_i - a_{i+1} \in T.$$

So a_i and $a_{i+1} = a_i + 1$ are adjacent. Therefore,

$$a \sim a_1 \sim a_2 \sim \cdots \sim a_k = b$$

is a path from a to b in $\mathrm{Tri}(\mathbb{Z}_n)$. Since a and b were arbitrary, the graph $\mathrm{Tri}(\mathbb{Z}_n)$ is connected. $\square$

Proposition 3.2. *Let $R = \mathbb{Z}_{p^a}$, where $p \geq 3$ and $a \geq 1$ or $p = 2^2$. Then $\mathrm{Tri}(R) = P_{p^a}$.*

Proof. Since $NT(p^a) = 1$, then by Lemma 3.2, we have two vertices of degree one and $p^a - 2$ vertices of degree 2. The proof of the connectedness is the same as Proposition 3.1. Hence $\mathrm{Tri}(R) = P_{p^a}$. $\square$

Proposition 3.3. *The girth of $Tri(\mathbb{Z}_n)$ is*

$$g(Tri(\mathbb{Z}_n)) = \begin{cases} \infty & n = p^a, p \text{ is odd prime}, a \geq 1 \text{ or } p = 2^2 \\ 4, & n = 2^a, a \geq 3, n = 2p^a, p \text{ is odd prime} \\ 3 & otherwise. \end{cases}$$

Proof. The proof of the girth to be ∞ follows from Proposition 3.2. By Lemma 3.1, we have a path from 0 to $n-1$, since $NT(\mathbb{Z}_n) > 1$, then there exist adjacent vertices a_i and a_j in $\mathbb{Z}_n$ such that $a_i - aj \neq n - 1$. From Lemma 3.1, part 2, we obtain the fourth cycle $a_i \sim a_j \sim a_j + 1 \sim a_i + 1$, which is the proof the second case. In the final case, the tri-potent set contains two elements which difference between them is 1, by Lemma 3.1, part 4. The result follows. $\square$

Proposition 3.4. *The tri-potent graph of $\mathbb{Z}_n$ has size $n(\frac{NT(n)+1}{2}) - 2^{m-1}$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$.*

Proof. The proof follows from Lemma 3.2 and the hand shaking lemma. $\square$

Proposition 3.5. *Let $R = \mathbb{Z}_{p^a}$ and $a \geq 2$. Then $\mathrm{Tri}(R) = P_{p^a}$ is Bipartite graph.*

Proof. Since p is odd prime and $p = 2$ with $a = 2$, then $NT(p^a) = 1$ and we obtain $\mathrm{Tri}(R) = P_{p^a}$. So we are done. Let $p = 2$ and $a \geq 3$, since $\mathbb{Z}_{p^a}$ is finite commutative ring with unity, then $\mathbb{Z}_{p^a} = Z(\mathbb{Z}_{p^a}) \cup U(\mathbb{Z}_{p^a})$ and $Tri(\mathbb{Z}_{p^a}) \subset U(\mathbb{Z}_{p^a})$. Take $V(Tri(\mathbb{Z}_{p^a})) = Z(\mathbb{Z}_{p^a}) \cup U(\mathbb{Z}_{p^a})$, from Lemma 3.1, we deduce that every vertex in $Z(\mathbb{Z}_{p^a})$ is adjacent to a vertex in $U(\mathbb{Z}_{p^a})$. However, the vertices in $Z(\mathbb{Z}_{p^a})$ (resp. $U(\mathbb{Z}_{p^a})$) are not adjacent because of their difference. But the difference in different components is an odd integer and it may or may not be tripotent element. Hence we are done. $\square$

Proposition 3.6. *Let $R = \mathbb{Z}_n$. Then*

1. $rad(Tri(R)) = diam(Tri(R)) = 2^{a-2}$, *if* $n = 2^a, a \geq 4$.
2. $rad(Tri(R)) = \frac{p^a+1}{2}$ *and* $diam(Tri(R)) = \frac{p^a+3}{2}$, *if* $n = 2p^a, p > 3$.
3. $rad(Tri(R)) = \frac{p^a-1}{2}$, *either* $rad(Tri(R)) = diam(Tri(R))$ *or* $diam(Tri(R)) = rad(Tri(R)) + 1$, *if* $n = 4p^a, p \neq 5$ *or* $n = 8p^a, p \geq 5$.

4. $rad(Tri(R)) = \frac{m-1}{2}$, *either* $rad(Tri(R)) = diam(Tri(R)) = \frac{m-1}{2}$ *or* $diam(Tri(R)) = rad(Tri(R)) + 1$, *where* $m = max\{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_r^{\alpha_r}\}$, *if* $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, *where* $p_i^{\alpha_i}$ *are distinct odd prime numbers.*

Proof. By Proposition 3.5, $Tri(\mathbb{Z}_{2^a})$ is bi partite graph and by Lemma 3.2, we have bi-regular tripotent graph, this implies that $d(a, b) \leq 2^{a-2}$ for all $a, b \in \mathbb{Z}_{2^a}$. Thus $rad(Tri(R)) = diam(Tri(R)) = 2^{a-2}$. The proof of the rest are similar. □

We end up with the following question: ”what does the tripotent graph of $H(\mathbb{F}_q)$ look like for any q?”

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