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Totally antimagic total labeling of helm and gear graphs

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Abstract

A total labeling of a graph G is a bijection from the union of the vertex set and the edge set of G to the set $\{1, 2, \ldots, |V(G)| + |E(G)|\}$. Under a total labeling, the vertex-weight of a vertex is defined as the sum of its label and the labels of all edges incident to it. Similarly, the edge-weight of an edge is the sum of its label and the labels of its two end vertices. A total labeling is said to be edge-antimagic total if all the edge-weights are pairwise distinct, and vertex-antimagic total if all the vertex-weights are pairwise distinct. If a total labeling is edge-antimagic total and vertex-antimagic total at the same time, then it is called a totally antimagic total labeling. A graph that admits a totally antimagic total labeling is called a totally antimagic total graph. In this paper, we show that helm graphs H_n and gear graphs G_n are totally antimagic total graphs.

Keywords: Totally antimagic total labeling, helm graph, gear graph
Mathematics Subject Classification : 05C78

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1. Introduction

In this study, we consider simple, finite, and undirected graphs. A graph G consists of a vertex-set $V(G)$ and an edge-set $E(G)$ such that each edge in $E(G)$ joins two distinct vertices in $V(G)$ and every two distinct vertices in $V(G)$ are joined by at most one edge in $E(G)$. The order of G is $|V(G)| = p$ and the size of G is $|E(G)| = q$.

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Two vertices $u, v \in V(G)$ is said to be adjacent if and only if there exists $e \in E(G)$ such that e joins u and v. Such an edge is commonly denoted by $e = uv$. The neighbor of $v \in V(G)$, denoted by $N(v)$, is the set of all vertices adjacent to v. That is, $N(V) = \{u \in V(G) \mid u \text{ is adjacent to } v\}$. A vertex $v \in V(G)$ is incident to an edge $e \in E(G)$ if and only if there exists $u \in V(G)$ such that $e = uv$. For more standard graph-theoretic terminology and notation, we follow [5].

A helm graph H_n is obtained by taking a wheel graph W_n and attaching a pendant vertex to each vertex on the rim. A gear graph G_n is constructed from a wheel graph W_n by inserting an additional vertex between every pair of adjacent vertices on the rim. Both graphs are related to the structure of a wheel, with helm graphs extending the cycle with pendant and gear graphs subdividing the cycle to resemble the teeth of a gear.

A labeling of a graph G is any mapping that sends a certain set of graph elements to a certain set of positive integers. The origins of modern graph labeling research can be traced to the concept of α, β, σ, and ρ labeling introduced by Rosa [15]. In the intervening years, more than 350 labeling techniques have emerged, giving rise to an extensive body of literature in graph theory comprising over 3,600 published papers [2].

Let f be a labeling of the graph G. If the domain of f is $V(G)$, then the labeling is called a vertex labeling. If the domain of f is $E(G)$, then the labeling is called an edge labeling. If the domain of f is $V(G) \cup E(G)$, then the labeling is called a total labeling. More precisely, for a graph G, a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ is a total labeling of G [4].

Under a total labeling f, the edge-weight of $uv \in E(G)$, denoted by $wt_f(uv)$, is defined by $wt_f(uv) = f(u) + f(uv) + f(v)$. On the other hand, the vertex-weight of $v \in V(G)$, denoted by $wt_f(v)$, is defined by $wt_f(v) = f(v) + \sum_{u \in N(v)} f(uv)$. In other words, the edge-weight of an edge is the sum of its label and the labels of its two end vertices. The vertex-weight of a vertex is defined as the sum of its label and the labels of all edges incident to it. For convenience, in this study, if $x \in V(G) \cup E(G)$, we will use $w(x)$ to denote the weight of x.

Hartsfield and Ringel [6] introduced the notion of an antimagic graph in 1990. A graph with q edges is called antimagic if its edges can be labeled with $1, 2, ..., q$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct. Hartsfield and Ringel proved that path $P_n (n \geq 3)$, cycle $C_n (n \geq 3)$, wheel $W_n (n \geq 3)$, and complete $K_n (n \geq 3)$ graphs are antimagic. Solairaju and Arockiasamy [7] showed that various families of subgraphs of grids $P_m \times P_n$ are antimagic. Shang et al. [8] proved that a star forest is antimagic if it contains no S_1 and at most one S_2 component. Hartsfield and Ringel [6] also conjectured that every tree except P_2 is antimagic and every connected graph except P_2 is antimagic. Alon et al. [9] used probabilistic methods and analytic number theory to show that this conjecture is true for all graphs with n vertices and minimum degree $\Omega(\log n)$.

Bača et al. [4] introduced the notion of a vertex-antimagic total labeling. A total labeling f of graph G is said to be a vertex-antimagic total labeling if and only if for all $u, v \in V(G)$, $wt_f(u) \neq wt_f(v)$. That is, the weights of all vertices under f are pairwise distinct. They also introduced the notion of (a, d)-vertex-antimagic total labeling. A total labeling f is a (a, d)-vertex-antimagic total labeling if and only if f is a vertex-antimagic total labeling and the weights of the vertices under f can be arranged in an arithmetic sequence where a is the first term and d is the common difference. They showed that paths and cycles admit (a, d)-vertex-antimagic total labelings for a wide variety of a and d. Bača et al. [4] studied (a, d)-vertex-antimagic total labelings of prisms, and generalized Petersen graphs. Lin et al. [14] proved that cycle C_n has (a, d)-vertex-antimagic total labeling if and only if $d = 0$ or $d = 2$ and n is odd, or $d = 1$. Ali et al. [13] showed the (a, d)-vertex antimagic total labelings of disjoint unions of regular graphs.

Simanjuntak et al. [10] introduced the notion of an edge-antimagic total labeling. A total labeling f of graph G is said to be an edge-antimagic total labeling if and only if for all $e, f \in E(G)$, $wt_f(e) \neq wt_f(f)$. That is, the weights of all edges under f are pairwise distinct. They also introduced the notion of (a, d)-edge-antimagic total labeling. A total labeling f is a (a, d)-edge-antimagic total labeling if and only if f is an edge-antimagic total labeling and the weights of the edges under f can be arranged in an arithmetic sequence where a is the first term and d is the common difference. They proved that cycle C_{2n+1} has a $(n+2, 1)$-edge-antimagic total labeling and a $(n + 3, 1)$-edge-antimagic total labeling, path P_{2n} has a $(n + 2, 1)$-edge-antimagic total labeling, and path P_n has a $(3, 2)$-edge-antimagic total labeling. Arumugam and Nalliah [11] studied the existence of (a, d)-edge antimagic total labelings for friendship graphs and generalized friendship graphs.

Bača et al. [3] introduced the notion of totally antimagic total labeling. A total labeling f of graph G is a totally antimagic total labeling if and only if f is a vertex-antimagic total labeling and an edge-antimagic total labeling at the same time. In addition, a graph that admits a totally antimagic total labeling is called a totally antimagic total graph. They proved that mK_1, mK_2, $P_n, n \geq 2$, and C_n admit a totally antimagic total labeling. Ahmed and Babujee [1] prove that complete bipartite graphs are totally antimagic total graphs. Ahmed et al. [12] prove that complete graphs admit a totally antimagic total labeling.

In this paper, we prove that helm H_n, and gear G_n graphs are totally antimagic total graphs.

2. Main Results

Theorem 2.1. *For every integer $n \geq 3$, the helm graph H_n admits a totally antimagic total labeling.*

Proof. Consider a helm graph H_n, $n \geq 3$, with $V(H_n) = \{v\} \cup \{v_i | i = 1, 2, ..., n\} \cup \{u_i | i = 1, 2, ..., n\}$, and $E(H_n) = \{vv_i | i = 1, 2, ..., n\} \cup \{v_i v_{i+1} | i = 1, 2, ..., n - 1\} \cup \{v_1 v_n\} \cup \{u_i v_i | i = 1, 2, ..., n\}$ where v is the central vertex, $v_i's$ are the rim vertices, $u_i's$ are the pendant vertices, $vv_i's$ are the spokes, $v_n v_1$ and $v_i v_{i+1}'s$ are the rim edges, and $u_i v_i's$ are the pendant edges. Also, $|V(H_n)| = 2n + 1$ and $|E(H_n)| = 3n$. Then, we define a function $f : V(H_n) \cup E(H_n) \rightarrow \{1, 2, 3, ..., 5n + 1\}$ such that:

$$
\begin{aligned}
f(v) &= 2n + 1, \quad 1 \leq i \leq n, \\
f(v_i) &= 4n - i + 2, \quad 1 \leq i \leq n, \\
f(u_i) &= i, \quad 1 \leq i \leq n \\
f(vv_i) &= 3n - i + 2, \quad 1 \leq i \leq n, \\
f(v_i v_{i+1}) &= 4n + i + 1, \quad 1 \leq i \leq n - 1, \\
f(v_n v_1) &= 5n + 1 \\
f(u_i v_i) &= n + i, \quad 1 \leq i \leq n
\end{aligned}
$$

The above-defined labeling f is a bijection. Thus, f is a total labeling.

Now, for the edge-weights:

- for $v_i v_{i+1}$, $1 \leq i \leq n - 1$:

$$
\begin{aligned}
w(vv_i) &= f(vv_i) + f(v) + f(v_i) \\
&= (3n - i + 2) + (2n + 1) + (4n - i + 2) \\
&= 9n - 2i + 5
\end{aligned}
$$

- for $v_i v_{i+1}$, $1 \leq i \leq n - 1$:

$$
\begin{aligned}
w(v_i v_{i+1}) &= f(v_i v_{i+1}) + f(v_i) + f(v_{i+1}) \\
&= (4n + i + 1) + (4n - i + 2) + (4n - i + 1) \\
&= 12n - i + 4
\end{aligned}
$$

- for $v_n v_1$:

$$
\begin{aligned}
w(v_n v_1) &= f(v_n v_1) + f(v_n) + f(v_1) \\
&= (5n + 1) + (3n + 2) + (4n + 1) \\
&= 12n + 4
\end{aligned}
$$

- for $u_i v_i$, $1 \leq i \leq n$:

$$
\begin{aligned}
w(u_i v_i) &= f(u_i v_i) + f(u_i) + f(v_i) \\
&= (n + i) + (i) + (4n - i + 2) \\
&= 5n + i + 2
\end{aligned}
$$

Notice that we can order the edge-weights increasingly such that

$$
w(u_1 v_1) < w(u_2 v_2) < \ldots < w(u_n v_n) < w(vv_n) < w(vv_{n-1}) < \ldots < w(vv_1)
$$

$$< w(v_{n-1}v_n) < w(v_{n-2}v_{n-1}) < \ldots < w(v_1v_2) < w(v_nv_1).$$

Thus, the edge-weights are pairwise distinct. Therefore, H_n, $n \geq 3$, admits an edge antimagic total labeling.

Now, for the vertex-weights:

- for v:

$$\begin{aligned} w(v) &= f(v) + \sum_{i=1}^{n} f(vv_i) \\ &= (2n+1) + \sum_{i=1}^{n}(3n-i+2) \\ &= \frac{5n^2+7n+2}{2} \end{aligned}$$

- for v_i, $2 \leq i \leq n-1$:

$$\begin{aligned} w(v_i) &= f(v_i) + f(vv_i) + f(u_iv_i) + f(v_{i-1}v_i) + f(v_iv_{i+1}) \\ &= (4n-i+2) + (3n-i+2) + (n+i) + (4n+i) + (4n+i+1) \\ &= 16n+i+5 \end{aligned}$$

- for v_1:

$$\begin{aligned} w(v_1) &= f(v_1) + f(vv_1) + f(u_1v_1) + f(v_nv_1) + f(v_1v_2) \\ &= (4n+1) + (3n+1) + (n+1) + (5n+1) + (4n+2) \\ &= 17n+6 \end{aligned}$$

- for v_n:

$$\begin{aligned} w(v_n) &= f(v_n) + f(vv_n) + f(u_nv_n) + f(v_{n-1}v_n) + f(v_nv_1) \\ &= (3n+2) + (2n+2) + (2n) + (5n) + (5n+1) \\ &= 17n+5 \end{aligned}$$

- for u_i, $1 \leq i \leq n$:

$$\begin{aligned} w(u_i) &= f(u_i) + f(u_iv_i) \\ &= (i) + (n+i) \\ &= n+2i \end{aligned}$$

For $3 \leq n \leq 5$, we can order the vertex-weights increasingly such that

$$w(u_1) < w(u_2) < \ldots < w(u_n) < w(v) < w(v_2) < w(v_3) < \ldots < w(v_{n-1}) < w(v_n) < w(v_1).$$

Also, for $n \geq 6$, we can order the vertex-weights increasingly such that

$$w(u_1) < w(u_2) < \ldots < w(u_n) < w(v_2) < w(v_3) < \ldots < w(v_{n-1}) < w(v_n) < w(v_1) < w(v).$$

Thus, the vertex-weights are pairwise distinct. Therefore, H_n, $n \geq 3$, admits a vertex antimagic total labeling.

Since f is simultaneously a vertex antimagic total labeling and an edge antimagic total labeling, then f is a totally antimagic total labeling. Therefore, for every integer $n \geq 3$, the helm graph H_n admits a totally antimagic total labeling, as desired. □

Now, to illustrate, we will consider the graph H_5 and H_8.

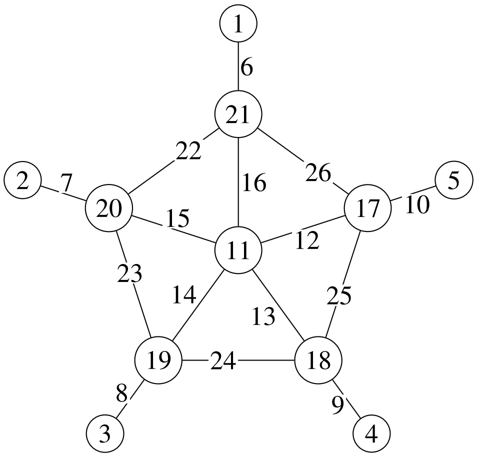

Figure 2.1. A totally antimagic total labeling of the graph H_5.

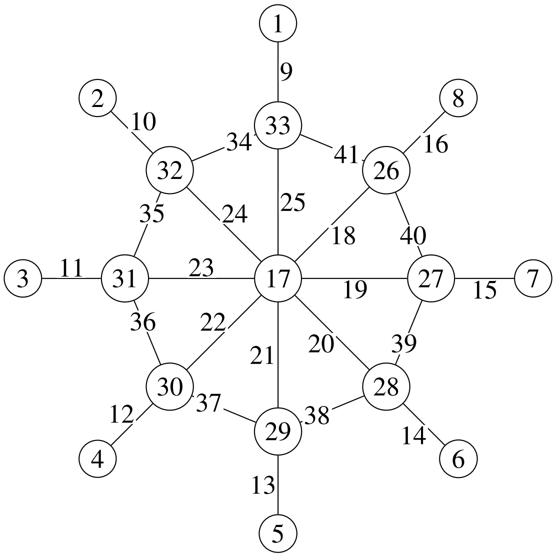

Figure 2.2. A totally antimagic total labeling of the graph H_8.

Theorem 2.2. *For every integer $n \geq 3$, the gear graph G_n admits a totally antimagic total labeling.*

Proof. Consider the gear graph G_n, $n \geq 3$, with $V(G_n) = \{v\} \cup \{v_i | i = 1, 2, .., n\} \cup \{u_i | i = 1, 2, ..., n\}$ and $E(G_n) = \{u_i v_i | i = 1, 2, ..., n\} \cup \{u_n v_1\} \cup \{u_i v_{i+1} | i = 1, 2, ..., n-1\} \cup \{v_i v | i = 1, 2, ..., n\}$ where v is the central vertex, $v_i's$ are the outer-cycle vertices, $u_i's$ are the subdivision vertices, $vv_i's$ are the spokes, $u_n v_1$, $u_i v_i's$, and $u_i v_{i+1}'s$ are the subdivision edges. Define the mapping $f : V(G_n) \cup E(G_n) \to \{1, 2, ..., 5n+1\}$ such that:

$$\begin{aligned}
f(v) &= 4n + 1\\
f(v_i) &= i, \quad 1 \leq i \leq n,\\
f(u_i) &= 2n + 1 - i, \quad 1 \leq i \leq n,\\
f(u_i v_i) &= 2n + 1 + i, \quad 1 \leq i \leq n,\\
f(u_n v_1) &= 2n + 1\\
f(u_i v_{i+1}) &= 4n + 1 - i, \quad 1 \leq i \leq n - 1,\\
f(v_i v) &= 4n + 1 + i, \quad 1 \leq i \leq n
\end{aligned}$$

The above-defined labeling f is a bijection. Hence, f is a total labeling.

Now, for the edge-weights:

- for $u_n v_1$:

$$\begin{aligned}
w(u_n v_1) &= f(u_n v_1) + f(u_n) + f(v_1)\\
&= (2n + 1) + (n + 1) + (1)\\
&= 3n + 3
\end{aligned}$$

- for $u_i v_{i+1}$, $1 \leq i \leq n - 1$:

$$\begin{aligned}
w(u_i v_{i+1}) &= f(u_i v_{i+1}) + f(u_i) + f(v_{i+1})\\
&= (4n - i + 1) + (2n - i + 1) + (i + 1)\\
&= 6n - i + 3
\end{aligned}$$

- for $u_i v_i$, $1 \leq i \leq n$:

$$\begin{aligned}
w(u_i v_i) &= f(u_i v_i) + f(u_i) + f(v_i)\\
&= (2n + i + 1) + (2n - i + 1) + (i)\\
&= 4n + i + 2
\end{aligned}$$

- for $v_i v$:

$$\begin{aligned} w(v_i v) &= f(v_i v) + f(v_i) + f(v) \\ &= (4n + i + 1) + (i) + (4n + 1) \\ &= 8n + 2i + 2 \end{aligned}$$

Notice that we can order the edge-weights increasingly such that

$$w(u_n v_1) < w(u_1 v_1) < w(u_2 v_2) < \ldots < w(u_n v_n) < w(u_{n-1} v_n) <$$
$$w(u_{n-2} v_{n-1}) < \ldots < w(u_1 v_2) < w(v_1 v) < w(v_2 v) < \ldots < w(v_n v).$$

Thus, the edge-weights are pairwise distinct. Therefore, G_n, $n \geq 3$, admits an edge antimagic total labeling.

Now, for the vertex-weights:

- for v_1:

$$\begin{aligned} w(v_1) &= f(v_1) + f(u_1 v_1) + f(u_n v_1) + f(v_1 v) \\ &= (1) + (2n + 2) + (2n + 1) + (4n + 2) \\ &= 8n + 6 \end{aligned}$$

- for v_i, $2 \leq i \leq n$:

$$\begin{aligned} w(v_i) &= f(v_i) + f(u_i v_i) + f(u_{i-1} v_i) + f(v_i v) \\ &= (i) + (2n + i + 1) + (4n - i + 2) + (4n + i + 1) \\ &= 10n + 2i + 4 \end{aligned}$$

- for u_i, $1 \leq i \leq n - 1$:

$$\begin{aligned} w(u_i) &= f(u_i) + f(u_i v_i) + f(u_i v_{i+1}) \\ &= (2n - i + 1) + (2n + i + 1) + (4n - i + 1) \\ &= 8n - i + 3 \end{aligned}$$

- for u_n:

$$\begin{aligned} w(u_n) &= f(u_n) + f(u_n v_n) + f(u_n v_1) \\ &= (n + 1) + (3n + 1) + (2n + 1) \\ &= 6n + 3 \end{aligned}$$

- for v:

$$\begin{aligned} w(v) &= f(v) + \sum_{i=1}^{n} f(v_i v) \\ &= (4n+1) + n(4n+1) + \frac{n(n+1)}{2} \\ &= \frac{9n^2 + 11n + 2}{2} \end{aligned}$$

Notice that we can order the vertex-weights increasingly such that

$$w(u_n) < w(u_{n-1}) < \ldots < w(u_1) < w(v_1) < w(v_2) < \ldots < w(v_n) < w(v).$$

Thus, the vertex-weights are pairwise distinct. Therefore, G_n, $n \geq 3$, admits an vertex antimagic total labeling.

Since f is simultaneously a vertex antimagic total labeling and an edge antimagic total labeling, then f is a totally antimagic total labeling. Therefore, for every integer $n \geq 3$, the gear graph G_n admits a totally antimagic total labeling, as desired. □

Now, to illustrate, we will consider the graph G_5.

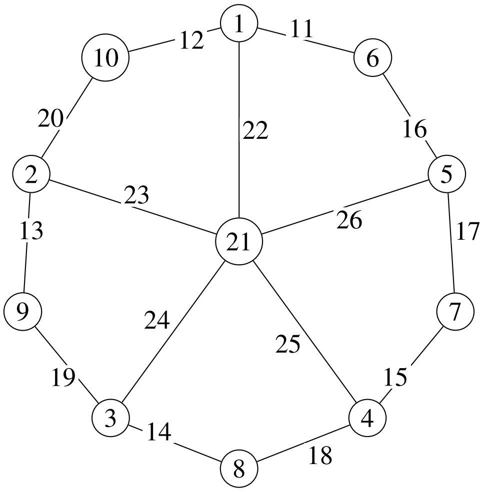

Figure 2.3. A totally antimagic total labeling of the graph G_5.

3. Conclusion

In this paper, we proved that helm graphs $H_n, n \geq 3$ and gear graphs $G_n, n \geq 3$ are totally antimagic total graphs. For further investigation we state the following open problems.

- Closed helm graphs, $CH_n, n \geq 3$, admit totally antimagic total labeling.
- Flower graphs, $FL_n, n \geq 3$, admit totally antimagic total labeling.

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