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The partition dimension of origami graphs and its barbell

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Abstract

The origami graph, $O_n, n \geq 3$, is a graph formed by a central cycle with origami folds, where each fold consists of two C_3 cycles. The barbell origami graph, B_{O_n} for $n \geq 3$ is obtained by copying a O_n and connecting two graphs with a bridge. In this research, we determined the partition dimension of the origami graphs and its barbell.

Keywords: origami graph, barbell origami graph, partition dimension
Mathematics Subject Classification : 05C12, 05C15

1. Introduction

The concept of partition dimension was introduced by Chartrand *et al.*[4]. Let $G = (V, E)$ be a connected graph with a vertex set $V(G)$ and a edge set $E(G)$. The distance vertex $v \in V(G)$ to the set $S \subseteq V(G)$, denoted by $d(v, S) = min\{d(v, x), x \in S\}$, where $d(v, x)$ is the length of shortest path v to x. Let $\Pi = \{S_1, S_2, S_3, \ldots, S_k\}$ be an ordered k-partition. The representation of v with respect to Π is defined as, $r(v|\Pi) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$. The partition Π is called a resolving partition of $V(G)$ if $r(v|\Pi) \neq r(u|\Pi)$ for any two distinct vertices $u, v \in V(G)$. The minimum k for which there is a resolving k-partition of $V(G)$ is the partition dimension of G, denoted by $pd(G)$ [4].

In general, determining the partition dimension of graph is a complex problem and newly interesting topic to study because there is no general theorem for determining the partition dimension of any graph. It is due to the various shapes and structures of the graphs. However, several criteria,

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constraints, and the partition dimension of certain graph classes have already been established. Chartrand *et al* [4] determined the partition dimension of paths and complete graphs. Nabila *et al*.[8] determined the partition dimension for edge amalgamation of Cycles, Rumarhobo *et al* [10] for umbrella graphs, Amrullah [1] for subdivision of homogeneous firecracker graphs, Asmiati *et al*.[2] for daisy graphs and its barbell, and Bhatti *et al*.[3] for generalized hexagonal cellular networks. Next, Hafidh *et al*.[5] analyzed partition dimension of trees-palm approach, Kuziak *et al*.[7] for edge partition dimension of graphs, whereas Ridwan *et al*.[9] for the dominating partition dimension.

This research investigates the partition dimension of origami graphs and barbell origami graphs. Let O_n, with $n \geq 3$, be an origami graph. The vertex set of O_n is $V(O_n) = \{u_i, v_i, w_i | 1 \leq i \leq n\}$ and its edge set is $E(O_n) = \{u_iw_i, u_iv_i, v_iw_i | 1 \leq i \leq n\} \cup \{u_iu_{i+1}, w_iu_{i+1} | 1 \leq i \leq n-1\} \cup \{u_1u_n, u_1w_n\}$. The barbell origami graph, B_{O_n}, for $n \geq 3$, is obtained by taking a copy of O_n, denoted by $O_n^{'}$, and connecting the two graphs by a bridge. The vertex set of $O_n^{'}$ is $\{u_i^{'}, v_i^{'}, w_i^{'} | 1 \leq i \leq n\}$ and the bridge is the edge $u_1u_1^{'}$ [6]. The following theorem is taken from Chartrand *et al*. [4].

Theorem 1.1. [4] *The partition dimension of cycle graph C_n for $n \geq 3$ is 3.*

2. Main Result

In this section, we determine the partition dimension of origami graphs and barbell origami graphs.

Theorem 2.1. *The partition dimension of origami graphs O_n for $n \geq 3$ is 3.*

Proof. The origami graph, O_n contains some cycle graph, then by Theorem 1.1, we have $pd(O_n) \geq 3$. To show the upper bound of $pd(O_n), n \geq 3$, we divided some cases.

Cases 1. $n \geq 3$ and $n \equiv 0(mod 3)$.
Given $\Pi = \{S_1, S_2, S_3\}$ with the partition is:
$S_1 = \{u_i, v_i, w_i | 1 \leq i \leq \frac{n}{3}\}$;
$S_2 = \{u_i, v_i, w_i | \frac{n+3}{3} \leq i \leq \frac{2n}{3}\}$;
$S_3 = \{u_i, v_i, w_i | \frac{2n+3}{3} \leq i \leq n\}$.

The representation of all vertices of O_n.

- $r(u_i|\Pi) = (0, \frac{n+3}{3} - i, i)$, for $1 \leq i \leq \frac{n}{3}$
- $r(u_i|\Pi) = (i - \frac{n}{3}, 0, \frac{2n+3}{3} - i)$, for $\frac{n+3}{3} \leq i \leq \frac{2n}{3}$
- $r(u_i|\Pi) = (n - i + 1, i - \frac{2n}{3}, 0)$, for $\frac{2n+3}{3} \leq i \leq n$
- $r(v_i|\Pi) = (0, \frac{n+3}{3} - i + 1, i + 1)$, for $1 \leq i \leq \frac{n}{3}$
- $r(v_i|\Pi) = (i - \frac{n}{3} + 1, 0, \frac{2n+3}{3} - i + 1)$, for $\frac{n+3}{3} \leq i \leq \frac{2n}{3}$

- $r(v_i|\Pi) = (n-i+2, i-\frac{2n}{3}+1, 0)$, for $\frac{2n+3}{3} \leq i \leq n$
- $r(w_i|\Pi) = (0, \frac{n+3}{3}-i, i+1)$, for $1 \leq i \leq \frac{n}{3}$
- $r(w_i|\Pi) = (i-\frac{n+3}{3}+1, 0, \frac{2n+3}{3}-i)$, for $\frac{n+3}{3} \leq i \leq \frac{2n}{3}$
- $r(w_i|\Pi) = (n-i+1, i-\frac{2n}{3}+1, 0)$, for $\frac{2n+3}{3} \leq i \leq n$

Case 2. $n \geq 3$ and $n \equiv 1(mod 3)$.
Given $\Pi = \{S_1, S_2, S_3\}$ with partition:
$S_1 = \{u_i|1 \leq i \leq \frac{n+2}{3}\} \cup \{v_i, w_i|1 \leq i \leq \frac{n-1}{3}\}$;
$S_2 = \{u_i|\frac{n+5}{3} \leq i \leq \frac{2n+1}{3}\} \cup \{v_i, w_i|\frac{n+2}{3} \leq i \leq \frac{2n+1}{3}\}$;
$S_3 = \{u_i, v_i, w_i|\frac{2n+4}{3} \leq i \leq n\}$

The representation of all vertices of O_n.

- $r(u_i|\Pi) = (0, \frac{n+5}{3}-i, i)$, for $1 \leq i \leq \frac{n+2}{3}$
- $r(u_i|\Pi) = (i-\frac{n+2}{3}, 0, \frac{2n+4}{3}-i)$, for $\frac{n+5}{3} \leq i \leq \frac{2n+1}{3}$
- $r(u_i|\Pi) = (n-i+1, i-\frac{2n+1}{3}, 0)$, for $\frac{2n+4}{3} \leq i \leq n$
- $r(v_i|\Pi) = (0, \frac{n+5}{3}-i+1, i+1)$, for $1 \leq i \leq \frac{n-1}{3}$
- $r(v_i|\Pi) = (i-\frac{n+2}{3}+1, 0, \frac{2n+4}{3}-i+1)$, for $\frac{n+2}{3} \leq i \leq \frac{2n+1}{3}$
- $r(v_i|\Pi) = (n-i+2, i-\frac{2n+1}{3}+1, 0)$, for $\frac{2n+4}{3} \leq i \leq n$
- $r(w_i|\Pi) = (0, \frac{n+5}{3}-i, i+1)$, for $1 \leq i \leq \frac{n-1}{3}$
- $r(w_i|\Pi) = (i-\frac{n+2}{3}+1, 0, \frac{2n+4}{3}-i)$, for $\frac{n+2}{3} \leq i \leq \frac{2n+1}{3}$
- $r(w_i|\Pi) = (n-i+1, i-\frac{2n+1}{3}+1, 0)$, for $\frac{2n+4}{3} \leq i \leq n$

Case 3. $n \geq 3$ and $n \equiv 2(mod 3)$.
Given $\Pi = \{S_1, S_2, S_3\}$ with partition:
$S_1 = \{u_i, v_i, w_i|1 \leq i \leq \frac{n+1}{3}\}$;
$S_2 = \{u_i|\frac{n+4}{3} \leq i \leq \frac{2n+2}{3}\} \cup \{v_i, w_i|\frac{n+4}{3} \leq i \leq \frac{2n-1}{3}\}$;
$S_3 = \{u_i|\frac{2n+5}{3} \leq i \leq n\} \cup \{v_i, w_i|\frac{2n+2}{3} \leq i \leq n\}$.

The representation of all vertices of O_n.

- $r(u_i|\Pi) = (0, \frac{n+4}{3}-i, i)$, for $1 \leq i \leq \frac{n+1}{3}$
- $r(u_i|\Pi) = (i-\frac{n+1}{3}, 0, \frac{2n+5}{3}-i)$, for $\frac{n+4}{3} \leq i \leq \frac{2n+2}{3}$
- $r(u_i|\Pi) = (n-i+1, i-\frac{2n+2}{3}, 0)$, for $\frac{2n+5}{3} \leq i \leq n$

- $r(v_i|\Pi) = (0, \frac{n+4}{3} - i + 1, i + 1)$, for $1 \leq i \leq \frac{n+1}{3}$
- $r(v_i|\Pi) = (i - \frac{n+1}{3} + 1, 0, \frac{2n+5}{3} - i + 1)$, for $\frac{n+4}{3} \leq i \leq \frac{2n-1}{3}$
- $r(v_i|\Pi) = (n - i + 2, i - \frac{2n+2}{3} + 1, 0)$, for $\frac{2n+2}{3} \leq i \leq n$
- $r(w_i|\Pi) = (0, \frac{n+4}{3} - i, i + 1)$, for $1 \leq i \leq \frac{n+1}{3}$
- $r(w_i|\Pi) = (i - \frac{n+1}{3} + 1, 0, \frac{2n+5}{3} - i)$, for $\frac{n+4}{3} \leq i \leq \frac{2n-1}{3}$
- $r(w_i|\Pi) = (n - i + 1, i - \frac{2n+2}{3} + 1, 0)$, for $\frac{2n+2}{3} \leq i \leq n$

Since the representations of all vertices are distinct, Π is a resolving partition of O_n for $n \geq 3$. Hence, $pd(O_n) \leq 3$. Therefore, $pd(O_n) = 3$. $\square$

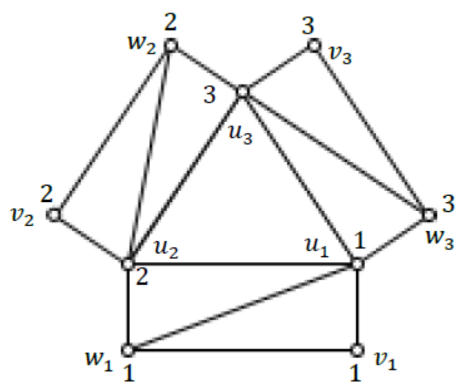

Figure 1. A minimum partition dimension of O_3.

Theorem 2.2. *The partition dimension of barbell origami graphs, B_{O_n} for $n \geq 3$ is 4.*

Proof. First, we determine the lower bound of $pd(Bo_n)$ for $n \geq 3$. Since a barbell origami graph contains two origami graphs, by Theorem 2.1 we have $pd(Bo_n) \geq 3$. Next, we will show that 3 partitions are not enough. For a contradiction, assume that there are 3 partitions in Bo_n for $n \geq 3$. A Barbell Origami graph, Bo_n contains $2n$ pieces of C_3. As a result, if we use only 3 partitions, then there are two vertices have the same representation, a contradiction.

Next, we determine the upper bound for the partition dimension of barbell Origami graph $pd(Bo_n)$ for $n \geq 3$ with some cases.
Cases 1. $n \geq 3$ and $n \equiv 0(mod3)$.

- $n = 3$.

  Given $\Pi = \{S_1, S_2, S_3, S_4\}$ with partition of Bo_3:

  $S_1 =\{u_1\} \cup \{v_1\} \cup \{w_1\} \cup \{u_1'\} \cup \{v_1'\} \cup \{w_1'\}$;

  $S_2 =\{u_2\} \cup \{v_2\} \cup \{w_2\} \cup \{u_2'\} \cup \{v_2'\} \cup \{w_2'\}$;

$S_3 =\{u_3\} \cup \{v_3\} \cup \{w_3\}$;
$S_4 =\{u_3^{'}\} \cup \{v_3^{'}\} \cup \{w_3^{'}\}$.

- $n > 3$
  Given $\Pi = \{S_1, S_2, S_3, S_4\}$ with partition of Bo_n for $n > 3$
  $S_1 =\{u_i, v_i, w_i, u_i^{'}, v_i^{'}, w_i^{'} | 1 \leq i \leq \frac{n-3}{3}\} \cup \{u_n, v_n, w_n, u_n^{'}, v_n^{'}, w_n^{'}\}$;
  $S_2 =\{u_i, v_i, w_i, u_i^{'}, v_i^{'}, w_i^{'} | \frac{n}{3} \leq i \leq \frac{2n-3}{3}\}$;
  $S_3 =\{u_i, v_i, w_i | \frac{2n}{3} \leq i \leq \frac{3n-3}{3}\}$;
  $S_4 =\{u_i^{'}, v_i^{'}, w_i^{'} | \frac{2n}{3} \leq i \leq \frac{3n-3}{3}\}$.

  The representation of all vertices is:

  - $r(u_i|\Pi) = (0, \frac{n}{3} - i, i + 1, i + 2)$, for $1 \leq i \leq \frac{n-3}{3}$
  - $r(u_i|\Pi) = (i - \frac{n-3}{3}, 0, \frac{2n}{3} - i, i + 2)$, for $\frac{n}{3} \leq i \leq \frac{2n-3}{3}, 6 \leq n \leq 12$
  - $r(u_i|\Pi) = (i - \frac{n-3}{3}, 0, \frac{2n}{3} - i, i + 2)$, for $\frac{n}{3} \leq i \leq \frac{n+1}{2}, n \geq 15$ odd
  - $r(u_i|\Pi) = (i - \frac{n-3}{3}, 0, \frac{2n}{3} - i, n - i + 4)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-3}{3}, n \geq 15$ odd
  - $r(u_i|\Pi) = (i - \frac{n-3}{3}, 0, \frac{2n}{3} - i, i + 2)$, for $\frac{n}{3} \leq i \leq \frac{n+2}{2}, n \geq 18$ even
  - $r(u_i|\Pi) = (i - \frac{n-3}{3}, 0, \frac{2n}{3} - i, n - i + 4)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-3}{3}, n \geq 18$ even
  - $r(u_i|\Pi) = (n - i, i - \frac{2n-3}{3}, 0, n - i + 4)$, for $\frac{2n}{3} \leq i \leq \frac{3n-3}{3}$
  - $r(u_n|\Pi) = (0, \frac{n}{3}, 1, 4)$.
  - $r(v_i|\Pi) = (0, \frac{n}{3} - i + 1, i + 2, i + 3)$, for $1 \leq i \leq \frac{n-3}{3}$
  - $r(v_i|\Pi) = (i - \frac{n-3}{3} + 1, 0, \frac{2n}{3} - i + 1, i + 3)$, for $\frac{n}{3} \leq i \leq \frac{2n-3}{3}, 6 \leq n \leq 12$
  - $r(v_i|\Pi) = (i - \frac{n-3}{3} + 1, 0, \frac{2n}{3} - i + 1, i + 3)$, for $\frac{n}{3} \leq i \leq \frac{n+1}{2}, n \geq 15$ odd
  - $r(v_i|\Pi) = (i - \frac{n-3}{3} + 1, 0, \frac{2n}{3} - i + 1, n - i + 5)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-3}{3}, n \geq 15$ odd
  - $r(v_i|\Pi) = (i - \frac{n-3}{3} + 1, 0, \frac{2n}{3} - i + 1, i + 3)$, for $\frac{n}{3} \leq i \leq \frac{n+2}{2}, n \geq 18$ even
  - $r(v_i|\Pi) = (i - \frac{n-3}{3} + 1, 0, \frac{2n}{3} - i + 1, n - i + 5)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-3}{3}, n \geq 18$ even
  - $r(v_i|\Pi) = (n - i + 1, i - \frac{2n-3}{3} + 1, 0, n - i + 5)$, for $\frac{2n}{3} \leq i \leq \frac{3n-3}{3}$
  - $r(v_n|\Pi) = (0, \frac{n}{3} + 1, 2, 5)$.
  - $r(w_i|\Pi) = (0, \frac{n}{3} - i, i + 2, i + 3)$, for $1 \leq i \leq \frac{n-3}{3}$
  - $r(w_i|\Pi) = (i - \frac{n-3}{3} + 1, 0, \frac{2n}{3} - i, i + 3)$, for $\frac{n}{3} \leq i \leq \frac{2n-3}{3}, n = 6$ and 9
  - $r(w_i|\Pi) = (i - \frac{n-3}{3} + 1, 0, \frac{2n}{3} - i, i + 3)$, for $\frac{n}{3} \leq i \leq \frac{n+1}{2}, n \geq 15$ odd
  - $r(w_i|\Pi) = (i - \frac{n-3}{3} + 1, 0, \frac{2n}{3} - i, n - i + 4)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-3}{3}, n \geq 15$ odd
  - $r(w_i|\Pi) = (i - \frac{n-3}{3} + 1, 0, \frac{2n}{3} - i, i + 3)$, for $\frac{n}{3} \leq i \leq \frac{n}{2}, n \geq 12$ even

- $r(w_i|\Pi) = (i - \frac{n-3}{3}, 0, \frac{2n}{3} - i, n - i + 4)$, for $\frac{n+2}{2} \leq i \leq \frac{2n-3}{3}, n \geq 12$ even
- $r(w_i|\Pi) = (n - i, i - \frac{2n-3}{3} + 1, 0, n - i + 4)$, for $\frac{2n}{3} \leq i \leq \frac{3n-3}{3}$
- $r(w_n|\Pi) = (0, \frac{n}{3}, 2, 4)$.
- $r(u_i^{'}|\Pi) = (0, \frac{n}{3} - i, i + 2, i + 1)$, for $1 \leq i \leq \frac{n-3}{3}$
- $r(u_i^{'}|\Pi) = (i - \frac{n-3}{3}, 0, i + 2, \frac{2n}{3} - i)$, for $\frac{n}{3} \leq i \leq \frac{2n-3}{3}, 6 \leq n \leq 12$
- $r(u_i^{'}|\Pi) = (i - \frac{n-3}{3}, 0, i + 2, \frac{2n}{3} - i)$, for $\frac{n}{3} \leq i \leq \frac{n+1}{2}, n \geq 15$ odd
- $r(u_i^{'}|\Pi) = (i - \frac{n-3}{3}, 0, n - i + 4, \frac{2n}{3} - i)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-3}{3}, n \geq 15$ odd
- $r(u_i^{'}|\Pi) = (i - \frac{n-3}{3}, 0, i + 2, \frac{2n}{3} - i)$, for $\frac{n}{3} \leq i \leq \frac{n+2}{2}, n \geq 18$ even
- $r(u_i^{'}|\Pi) = (i - \frac{n-3}{3}, 0, n - i + 4, \frac{2n}{3} - i)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-3}{3}, n \geq 18$ even
- $r(u_i^{'}|\Pi) = (n - i, i - \frac{2n-3}{3}, n - i + 4, 0)$, for $\frac{2n}{3} \leq i \leq \frac{3n-3}{3}$
- $r(u_n^{'}|\Pi) = (0, \frac{n}{3}, 4, 1)$.
- $r(v_i^{'}|\Pi) = (0, \frac{n}{3} - i + 1, i + 3, i + 2)$, for $1 \leq i \leq \frac{n-3}{3}$
- $r(v_i^{'}|\Pi) = (i - \frac{n-3}{3} + 1, 0, i + 3, \frac{2n}{3} - i + 1)$, for $\frac{n}{3} \leq i \leq \frac{2n-3}{3}, 6 \leq n \leq 12$
- $r(v_i^{'}|\Pi) = (i - \frac{n-3}{3} + 1, 0, i + 3, \frac{2n}{3} - i + 1)$, for $\frac{n}{3} \leq i \leq \frac{n+1}{2}, n \geq 15$ odd
- $r(v_i^{'}|\Pi) = (i - \frac{n-3}{3} + 1, 0, n - i + 5, \frac{2n}{3} - i + 1)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-3}{3}, n \geq 15$ odd
- $r(v_i^{'}|\Pi) = (i - \frac{n-3}{3} + 1, 0, i + 3, \frac{2n}{3} - i + 1)$, for $\frac{n}{3} \leq i \leq \frac{n+2}{2}, n \geq 18$ even
- $r(v_i^{'}|\Pi) = (i - \frac{n-3}{3} + 1, 0, n - i + 5, \frac{2n}{3} - i + 1)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-3}{3}, n \geq 18$ even
- $r(v_i^{'}|\Pi) = (n - i + 1, i - \frac{2n-3}{3} + 1, n - i + 5, 0)$, for $\frac{2n}{3} \leq i \leq \frac{3n-3}{3}$
- $r(v_n^{'}|\Pi) = (0, \frac{n}{3} + 1, 5, 2)$.
- $r(w_i^{'}|\Pi) = (0, \frac{n}{3}, 4, 2)$, for $i = 1$
- $r(w_i^{'}|\Pi) = (0, \frac{n}{3} - i + 1, i + 2, i + 1)$, for $2 \leq i \leq \frac{n-3}{3}, n \geq 9$
- $r(w_i^{'}|\Pi) = (i - \frac{n-3}{3}, 0, i + 2, \frac{2n}{3} - i + 1)$, for $\frac{n}{3} \leq i \leq \frac{2n-3}{3}, 6 \leq n \leq 15$
- $r(w_i^{'}|\Pi) = (i - \frac{n-3}{3}, 0, i + 2, \frac{2n}{3} - i + 1)$, for $\frac{n}{3} \leq i \leq \frac{n+3}{2}, n \geq 21$ odd
- $r(w_i^{'}|\Pi) = (i - \frac{n-3}{3}, n - i + 5, \frac{2n}{3} - i + 1)$, for $\frac{n+5}{2} \leq i \leq \frac{2n-3}{3}, n \geq 21$ odd
- $r(w_i^{'}|\Pi) = (i - \frac{n-3}{3}, 0, i + 2, \frac{2n}{3} - i + 1)$, for $\frac{n}{3} \leq i \leq \frac{n+2}{2}, n \geq 18$ even
- $r(w_i^{'}|\Pi) = (i - \frac{n-3}{3}, n - i + 5, \frac{2n}{3} - i + 1)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-3}{3}, n \geq 18$ even
- $r(w_i^{'}|\Pi) = (n - i + 1, i - \frac{2n-3}{3}, 6, 0)$, for $\frac{2n}{3} \leq i \leq \frac{3n-3}{3}, n = 6$
- $r(w_i^{'}|\Pi) = (n - i + 1, i - \frac{2n-3}{3}, n - i + 5, 0)$, for $\frac{2n}{3} \leq i \leq \frac{3n-3}{3}, n \geq 9$
- $r(w_n^{'}|\Pi) = (0, \frac{n}{3} + 1, 5, 1)$.

Case 2. $n \geq 4$ and $n \equiv 1(mod3)$.

- for $n = 4$.
  The given $\Pi = \{S_1, S_2, S_3, S_4\}$ with partition of Bo_4.
  $S_1 = \{u_1, u_4, v_4, w_4, u_1', u_4', v_1', v_4', w_1', w_4'\}$;
  $S_2 = \{u_2, u_2', v_2', w_2'\} \cup \{v_i, w_i | 1 \leq i \leq \frac{n+2}{3}\}$;
  $S_3 = \{u_3, v_3, w_3\}$;
  $S_4 = \{u_3', v_3', w_3'\}$.

  Clearly, all vertices have different representation.

- $n > 4$
  Given $\Pi = \{S_1, S_2, S_3, S_4\}$ with partition of Bo_n for $n > 4$.
  $S_1 = \{u_i, v_i, w_i, u_i', v_i', w_i' | 1 \leq i \leq \frac{n-1}{3}\} \cup \{u_n, v_n, w_n, u_n', v_n', w_n'\}$;
  $S_2 = \{u_i, u_i', v_i', w_i' | \frac{n+2}{3} \leq i \leq \frac{2n-2}{3}\} \cup \{v_i, w_i | \frac{n-1}{3} \leq i \leq \frac{2n-2}{3}\}$;
  $S_3 = \{u_i, v_i, w_i | \frac{2n+1}{3} \leq i \leq \frac{3n-3}{3}\}$;
  $S_4 = \{u_i', v_i', w_i' | \frac{2n+1}{3} \leq i \leq \frac{3n-3}{3}\}$.

  The representation of all vertices is:
  * $r(u_i|\Pi) = (0, \frac{n+2}{3} - i, i+1, i+2)$, for $1 \leq i \leq \frac{n-1}{3}$
  * $r(u_i|\Pi) = (i - \frac{n-1}{3}, 0, \frac{2n+1}{3} - i, i+2)$, for $\frac{n+2}{3} \leq i \leq \frac{2n-2}{3}, n = 7$ and 10
  * $r(u_i|\Pi) = (i - \frac{n-1}{3}, 0, \frac{2n+1}{3} - i, i+2)$, for $\frac{n+2}{3} \leq i \leq \frac{n+1}{2}, n \geq 13$ odd
  * $r(u_i|\Pi) = (i - \frac{n-1}{3}, 0, \frac{2n+1}{3} - i, n-i+4)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-2}{3}, n \geq 13$ odd
  * $r(u_i|\Pi) = (i - \frac{n-1}{3}, 0, \frac{2n+1}{3} - i, i+2)$, for $\frac{n+2}{3} \leq i \leq \frac{n+2}{2}, n \geq 16$ even
  * $r(u_i|\Pi) = (i - \frac{n-1}{3}, 0, \frac{2n+1}{3} - i, n-i+4)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-2}{3}, n \geq 16$ even
  * $r(u_i|\Pi) = (n-i, i - \frac{2n-3}{3}, 0, n-i+4)$, for $\frac{2n+1}{3} \leq i \leq \frac{3n-3}{3}$
  * $r(u_n|\Pi) = (0, \frac{n+2}{3}, 1, 4)$.
  * $r(v_i|\Pi) = (0, \frac{n+2}{3} - i + 1, i+2, i+3)$, for $1 \leq i \leq \frac{n-4}{3}$
  * $r(v_i|\Pi) = (i - \frac{n-1}{3} + 1, 0, \frac{2n+1}{3} - i + 1, i+3)$, for $\frac{n-1}{3} \leq i \leq \frac{2n-2}{3}, n = 7$ and 10
  * $r(v_i|\Pi) = (i - \frac{n-1}{3} + 1, 0, \frac{2n+1}{3} - i + 1, i+3)$, for $\frac{n-1}{3} \leq i \leq \frac{n+1}{2}, n \geq 13$ odd
  * $r(v_i|\Pi) = (i - \frac{n-1}{3} + 1, 0, \frac{2n+1}{3} - i + 1, n-i+5)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-2}{3}, n \geq 13$ odd
  * $r(v_i|\Pi) = (i - \frac{n-1}{3} + 1, 0, \frac{2n+1}{3} - i + 1, i+3)$, for $\frac{n-1}{3} \leq i \leq \frac{n+2}{2}, n \geq 16$ even
  * $r(v_i|\Pi) = (i - \frac{n-1}{3} + 1, 0, \frac{2n+1}{3} - i + 1, n-i+5)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-2}{3}, n \geq 16$ even
  * $r(v_i|\Pi) = (n-i+1, i - \frac{2n-2}{3} + 1, 0, n-i+5)$, for $\frac{2n+1}{3} \leq i \leq \frac{3n-3}{3}$
  * $r(v_n|\Pi) = (0, \frac{n+2}{3} + 1, 2, 5)$.
  * $r(w_i|\Pi) = (0, \frac{n+2}{3} - i, i+2, i+3)$, for $1 \leq i \leq \frac{n-4}{3}$
  * $r(w_i|\Pi) = (i - \frac{n-1}{3} + 1, 0, \frac{2n+1}{3} - i, i+3)$, for $\frac{n-1}{3} \leq i \leq \frac{2n-2}{3}, n = 6$

* $r(w_i|\Pi) = (i - \frac{n-1}{3} + 1, 0, \frac{2n+1}{3} - i, i+3)$, for $\frac{n-1}{3} \leq i \leq \frac{n+1}{2}, n \geq 13$ odd
* $r(w_i|\Pi) = (i - \frac{n-1}{3} + 1, 0, \frac{2n+1}{3} - i, n-i+4)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-2}{3}, n \geq 13$ odd
* $r(w_i|\Pi) = (i - \frac{n-1}{3} + 1, 0, \frac{2n+1}{3} - i, i+3)$, for $\frac{n-1}{3} \leq i \leq \frac{n}{2}, n \geq 10$ even
* $r(w_i|\Pi) = (i - \frac{n-3}{3}, 0, \frac{2n}{3} - i, n-i+4)$, for $\frac{n+2}{2} \leq i \leq \frac{2n-3}{3}, n \geq 10$ even
* $r(w_i|\Pi) = (n-i, i - \frac{2n-2}{3} + 1, 0, n-i+4)$, for $\frac{2n+1}{3} \leq i \leq \frac{3n-3}{3}$
* $r(w_n|\Pi) = (0, \frac{n+2}{3}, 2, 4)$.
* $r(u_i^{'}|\Pi) = (0, \frac{n+2}{3} - i, i+2, i+1)$, for $1 \leq i \leq \frac{n-1}{3}$
* $r(u_i^{'}|\Pi) = (i - \frac{n-1}{3}, 0, i+2, \frac{2n+1}{3} - i)$, for $\frac{n+2}{3} \leq i \leq \frac{2n-2}{3}, n = 7$ and 10
* $r(u_i^{'}|\Pi) = (i - \frac{n-1}{3}, 0, i+2, \frac{2n+1}{3} - i)$, for $\frac{n+2}{3} \leq i \leq \frac{n+1}{2}, n \geq 13$ odd
* $r(u_i^{'}|\Pi) = (i - \frac{n-1}{3}, 0, n-i+4, \frac{2n+1}{3} - i)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-2}{3}, n \geq 13$ odd
* $r(u_i^{'}|\Pi) = (i - \frac{n-1}{3}, 0, i+2, \frac{2n+1}{3} - i)$, for $\frac{n+2}{3} \leq i \leq \frac{n+2}{2}, n \geq 16$ even
* $r(u_i^{'}|\Pi) = (i - \frac{n-1}{3}, 0, n-i+4, \frac{2n+1}{3} - i)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-2}{3}, n \geq 16$ even
* $r(u_i^{'}|\Pi) = (n-i, i - \frac{2n-2}{3}, n-i+4, 0)$, for $\frac{2n+1}{3} \leq i \leq \frac{3n-3}{3}$
* $r(u_n^{'}|\Pi) = (0, \frac{n+2}{3}, 4, 1)$.
* $r(v_i^{'}|\Pi) = (0, \frac{n+2}{3} - i + 1, i+3, i+2)$, for $1 \leq i \leq \frac{n-1}{3}$
* $r(v_i^{'}|\Pi) = (i - \frac{n-1}{3} + 1, 0, i+3, \frac{2n+1}{3} - i + 1)$, for $\frac{n+2}{3} \leq i \leq \frac{2n-2}{3}, n = 7$ and 10
* $r(v_i^{'}|\Pi) = (i - \frac{n-1}{3} + 1, 0, i+3, \frac{2n+1}{3} - i + 1)$, for $\frac{n+2}{3} \leq i \leq \frac{n+1}{2}, n \geq 13$ odd
* $r(v_i^{'}|\Pi) = (i - \frac{n-1}{3} + 1, 0, n-i+5, \frac{2n+1}{3} - i + 1)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-2}{3}, n \geq 13$ odd
* $r(v_i^{'}|\Pi) = (i - \frac{n-1}{3} + 1, 0, i+3, \frac{2n+1}{3} - i + 1)$, for $\frac{n+2}{3} \leq i \leq \frac{n+2}{2}, n \geq 16$ even
* $r(v_i^{'}|\Pi) = (i - \frac{n-1}{3} + 1, 0, n-i+5, \frac{2n+1}{3} - i + 1)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-2}{3}, n \geq 16$ even
* $r(v_i^{'}|\Pi) = (n-i+1, i - \frac{2n-2}{3} + 1, n-i+5, 0)$, for $\frac{2n+1}{3} \leq i \leq \frac{3n-3}{3}$
* $r(v_n^{'}|\Pi) = (0, \frac{n+2}{3} + 1, 5, 2)$.
* $r(w_1^{'}|\Pi) = (0, \frac{n+2}{3}, 4, 2)$.
* $r(w_i^{'}|\Pi) = (0, \frac{n+2}{3} - i + 1, i+2, i+1)$, for $2 \leq i \leq \frac{n-1}{3}, n \geq 10$
* $r(w_i^{'}|\Pi) = (i - \frac{n-1}{3}, 0, i+2, \frac{2n+1}{3} - i + 1)$, for $\frac{n+2}{3} \leq i \leq \frac{2n-2}{3}, 7 \leq n \leq 13$
* $r(w_i^{'}|\Pi) = (i - \frac{n-1}{3}, 0, i+2, \frac{2n+1}{3} - i + 1)$, for $\frac{n+2}{3} \leq i \leq \frac{n+3}{2}, n \geq 19$ odd
* $r(w_i^{'}|\Pi) = (i - \frac{n-1}{3}, 0, n-i+5, \frac{2n+1}{3} - i + 1)$, for $\frac{n+5}{2} \leq i \leq \frac{2n-2}{3}, n \geq 19$ odd
* $r(w_i^{'}|\Pi) = (i - \frac{n-1}{3}, 0, i+2, \frac{2n+1}{3} - i + 1)$, for $\frac{n+2}{3} \leq i \leq \frac{n+2}{2}, n \geq 16$ even
* $r(w_i^{'}|\Pi) = (i - \frac{n-1}{3}, 0, n-i+5, \frac{2n+1}{3} - i + 1)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-2}{3}, n \geq 16$ even
* $r(w_i^{'}|\Pi) = (n-i+1, i - \frac{2n-2}{3}, n-i+5, 0)$, for $\frac{2n+1}{3} \leq i \leq \frac{3n-3}{3}$
* $r(w_n^{'}|\Pi) = (0, \frac{n+2}{3}, 5, 1)$.

Case 3. $n \geq 5$ and $n \equiv 2(mod 3)$.

- $n = 5$.

  Given $\Pi = \{S_1, S_2, S_3, S_4\}$ with partition of Bo_5 is:
  $S_1 = \{u_1, u_n, v_n, w_n, u_1^{'}, u_n^{'}, v_1^{'}, v_n^{'}, w_1^{'}, w_n^{'}\}$;
  $S_2 = \{u_{\frac{n+1}{3}}, u_{\frac{n+1}{3}}^{'}\} \cup \{v_i, w_i | 1 \leq i \leq \frac{n+1}{3}\} \cup \{v_i^{'}, w_i^{'} | \frac{n+1}{3} \leq i \leq \frac{n+4}{3}\}$;
  $S_3 = \{u_i, v_i, w_i | \frac{n+4}{3} \leq i \leq \frac{n+7}{3}\}$;
  $S_4 = \{u_i^{'} | \frac{n+4}{3} \leq i \leq \frac{n+7}{3}\} \cup \{v_{\frac{n+7}{3}}^{'}, w_{\frac{n+7}{3}}^{'}\}$.

  Then the representation of all vertices of the Bo_n graph is:

  * $r(u_1|\Pi) = (0, \frac{n+1}{3} - 1, 2, 3)$.
  * $r(u_2|\Pi) = (2 - \frac{n-2}{3}, 0, \frac{2n-1}{3} - 2, 4)$.
  * $r(u_i|\Pi) = (n - i, i - \frac{2n-4}{3}, 0, 5)$, for $i = 3$ and 4.
  * $r(u_5|\Pi) = (0, \frac{n+1}{3}, 1, 4)$.
  * $r(v_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i + 1, i + 3)$, for $i = 1$ and 2.
  * $r(v_i|\Pi) = (n - i + 1, i - \frac{2n-4}{3} + 1, 0, n - i + 5)$, for $i = 3$ and 4.
  * $r(v_5|\Pi) = (0, \frac{n-2}{3} + 2, 2, 5)$, for $i = 5$.
  * $r(w_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i, i + 3)$, for $i = 1$ and 2.
  * $r(w_i|\Pi) = (n - i, i - \frac{2n-4}{3} + 1, 0, n - i + 4)$, for $i = 3$ and 4.
  * $r(w_5|\Pi) = (0, \frac{n+1}{3}, 2, 4)$, for $i = 5$.
  * $r(u_1^{'}|\Pi) = (0, \frac{n+1}{3} - 1, 3, 2)$.
  * $r(u_2^{'}|\Pi) = (2 - \frac{n-2}{3}, 0, 4, \frac{2n-1}{3} - 2)$, for $i = 2$.
  * $r(u_i^{'}|\Pi) = (n - i, i - \frac{2n-4}{3}, n - i + 4, 0)$, for $i = 3$ and 4.
  * $r(u_5^{'}|\Pi) = (0, \frac{n+1}{3}, 4, 1)$.
  * $r(v_1^{'}|\Pi) = (0, \frac{n-2}{3} - 1, 4, 3)$.
  * $r(v_i^{'}|\Pi) = (i - \frac{n-2}{3} + 1, 0, i + 3, \frac{2n+2}{3} - i)$, for $i = 2$ and 3.
  * $r(v_4^{'}|\Pi) = (n - 3, 6 - \frac{2n-1}{3}, n + 1, 0)$.
  * $r(v_5^{'}|\Pi) = (0, \frac{n-2}{3} + 2, 5, 2)$.
  * $r(w_1^{'}|\Pi) = (0, \frac{n+1}{3}, 4, 2)$.
  * $r(w_i^{'}|\Pi) = (i - \frac{n-2}{3}, 0, i + 2, \frac{2n+2}{3} - i)$, for $i = 2$ and 3.
  * $r(w_4^{'}|\Pi) = (n - 3, 4 - \frac{2n-4}{3}, n - +1, 0)$.
  * $r(w_5^{'}|\Pi) = (0, \frac{n-2}{3} + 2, 5, 1)$,

- $n > 5$

  Given $\Pi = \{S_1, S_2, S_3, S_4\}$ with partition of Bo_n for $n > 5$ is: $S_1 = \{u_i, v_i, w_i, u_i^{'}, v_i^{'}, w_i^{'} | 1 \leq i \leq \frac{n-2}{3}\} \cup \{u_n, v_n, w_n, u_n^{'}, v_n^{'}, w_n^{'}\}$;
  $S_2 = \{u_i, u_i^{'}, v_i^{'}, w_i^{'} | \frac{n+1}{3} \leq i \leq \frac{2n-4}{3}\} \cup \{v_i, w_i | \frac{n-2}{3} \leq i \leq \frac{2n-4}{3}\}$;
  $S_3 = \{u_i, v_i, w_i | \frac{2n-1}{3} \leq i \leq \frac{3n-3}{3}\}$;

$S_4 = \{u_i^{'} | \frac{2n-1}{3} \leq i \leq \frac{3n-3}{3}\} \cup \{v_i^{'}, w_i^{'} | \frac{2n+2}{3} \leq i \leq \frac{3n-3}{3}\}$.

The representation of all vertices is:

* $r(u_i|\Pi) = (0, \frac{n+1}{3} - i, i+1, i+2)$, for $1 \leq i \leq \frac{n-2}{3}$
* $r(u_i|\Pi) = (i - \frac{n-2}{3}, 0, \frac{2n-1}{3} - i, i+2)$, for $\frac{n+1}{3} \leq i \leq \frac{2n-4}{3}, 8 \leq n \leq 14$
* $r(u_i|\Pi) = (i - \frac{n-2}{3}, 0, \frac{2n-1}{3} - i, i+2)$, for $\frac{n+1}{3} \leq i \leq \frac{n+1}{2}, n \geq 17$ odd
* $r(u_i|\Pi) = (i - \frac{n-2}{3}, 0, \frac{2n-1}{3} - i, n-i+4)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-4}{3}, n \geq 17$ odd
* $r(u_i|\Pi) = (i - \frac{n-2}{3}, 0, \frac{2n-1}{3} - i, i+2)$, for $\frac{n+1}{3} \leq i \leq \frac{n+2}{2}, n \geq 20$ even
* $r(u_i|\Pi) = (i - \frac{n-2}{3}, 0, \frac{2n-1}{3} - i, n-i+4)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-4}{3}, n \geq 20$ even
* $r(u_i|\Pi) = (n-i, i - \frac{2n-4}{3}, 0, n-i+4)$, for $\frac{2n-1}{3} \leq i \leq \frac{3n-3}{3}$
* $r(u_n|\Pi) = (0, \frac{n+1}{3}, 1, 4)$.
* $r(v_i|\Pi) = (0, \frac{n+2}{3} - i + 2, i+2, i+3)$, for $1 \leq i \leq \frac{n-5}{3}$
* $r(v_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i + 1, i+3)$, for $\frac{n-2}{3} \leq i \leq \frac{2n-4}{3}, 8 \leq n \leq 14$
* $r(v_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i + 1, i+3)$, for $\frac{n-2}{3} \leq i \leq \frac{n+1}{2}, n \geq 17$ odd
* $r(v_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i + 1, n-i+5)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-4}{3}, n \geq 17$ odd
* $r(v_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i + 1, i+3)$, for $\frac{n-2}{3} \leq i \leq \frac{n+2}{2}, n \geq 20$ even
* $r(v_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i + 1, n-i+5)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-4}{3}, n \geq 20$ even
* $r(v_i|\Pi) = (n-i+1, i - \frac{2n-4}{3} + 1, 0, n-i+5)$, for $\frac{2n-1}{3} \leq i \leq \frac{3n-3}{3}$
* $r(v_n|\Pi) = (0, \frac{n-2}{3} + 2, 2, 5)$.
* $r(w_i|\Pi) = (0, \frac{n+1}{3} - i, i+2, i+3)$, for $1 \leq i \leq \frac{n-5}{3}$
* $r(w_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i, i+3)$, for $\frac{n-2}{3} \leq i \leq \frac{2n-4}{3}, n = 8$ and 11
* $r(w_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i, i+3)$, for $\frac{n-2}{3} \leq i \leq \frac{n+1}{2}, n \geq 17$ odd
* $r(w_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i, n-i+4)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-4}{3}, n \geq 17$ odd
* $r(w_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i, i+3)$, for $\frac{n-2}{3} \leq i \leq \frac{n}{2}, n \geq 14$ even
* $r(w_i|\Pi) = (i - \frac{n-5}{3}, 0, \frac{2n-1}{3} - i, n-i+4)$, for $\frac{n+2}{2} \leq i \leq \frac{2n-4}{3}, n \geq 14$ even
* $r(w_i|\Pi) = (n-i, i - \frac{2n-4}{3} + 1, 0, n-i+4)$, for $\frac{2n-1}{3} \leq i \leq \frac{3n-3}{3}$
* $r(w_n|\Pi) = (0, \frac{n+1}{3}, 2, 4)$.
* $r(u_i^{'}|\Pi) = (0, \frac{n+1}{3} - i, i+2, i+1)$, for $1 \leq i \leq \frac{n-2}{3}$
* $r(u_i^{'}|\Pi) = (i - \frac{n-2}{3}, 0, i+2, \frac{2n-1}{3} - i)$, for $\frac{n+1}{3} \leq i \leq \frac{2n-4}{3}, 8 \leq n \leq 14$
* $r(u_i^{'}|\Pi) = (i - \frac{n-2}{3}, 0, i+2, \frac{2n-1}{3} - i)$, for $\frac{n+1}{3} \leq i \leq \frac{n+1}{2}, n \geq 17$ odd
* $r(u_i^{'}|\Pi) = (i - \frac{n-2}{3}, 0, n-i+4, \frac{2n-1}{3} - i)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-4}{3}, n \geq 17$ odd
* $r(u_i^{'}|\Pi) = (i - \frac{n-2}{3}, 0, i+2, \frac{2n-1}{3} - i)$, for $\frac{n+1}{3} \leq i \leq \frac{n+2}{2}, n \geq 20$ even
* $r(u_i^{'}|\Pi) = (i - \frac{n-2}{3}, 0, n-i+4, \frac{2n-1}{3} - i)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-4}{3}, n \geq 20$ even
* $r(u_i^{'}|\Pi) = (n-i, i - \frac{2n-4}{3}, n-i+4, 0)$, for $\frac{2n-1}{3} \leq i \leq \frac{3n-3}{3}$
* $r(u_n^{'}|\Pi) = (0, \frac{n+1}{3}, 4, 1)$.
* $r(v_i^{'}|\Pi) = (0, \frac{n-2}{3} - i + 2, i+3, i+2)$, for $1 \leq i \leq \frac{n-2}{3}$

* $r(v_i'|\Pi) = (i - \frac{n-2}{3} + 1, 0, i+3, \frac{2n+2}{3} - i)$, for $\frac{n+1}{3} \leq i \leq \frac{2n-1}{3}, n = 8$ and
* $r(v_i'|\Pi) = (i - \frac{n-2}{3} + 1, 0, i+3, \frac{2n+2}{3} - i)$, for $\frac{n+1}{3} \leq i \leq \frac{n+1}{2}, n \geq 11$ odd
* $r(v_i'|\Pi) = (i - \frac{n-2}{3} + 1, 0, n-i+5, \frac{2n+2}{3} - i)$, for $\frac{n+3}{2} \leq i \leq \frac{2n-1}{3}, n \geq 11$ odd
* $r(v_i'|\Pi) = (i - \frac{n-2}{3} + 1, 0, i+3, \frac{2n+2}{3} - i)$, for $\frac{n+1}{3} \leq i \leq \frac{n+2}{2}, n \geq 14$ even
* $r(v_i'|\Pi) = (i - \frac{n-2}{3} + 1, 0, n-i+5, \frac{2n+2}{3} - i)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-1}{3}, n \geq 14$ even
* $r(v_i'|\Pi) = (n-i+1, i - \frac{2n-1}{3} + 2, n-i+5, 0)$, for $\frac{2n+2}{3} \leq i \leq \frac{3n-3}{3}$
* $r(v_n'|\Pi) = (0, \frac{n-2}{3} + 2, 5, 2)$.
* $r(w_1'|\Pi) = (0, \frac{n+1}{3}, 4, 2)$.
* $r(w_i'|\Pi) = (0, \frac{n-2}{3} - i + 2, i+2, i+1)$, for $2 \leq i \leq \frac{n-2}{3}, n \geq 8$
* $r(w_i'|\Pi) = (i - \frac{n-2}{3}, 0, i+2, \frac{2n+2}{3} - i)$, for $\frac{n+1}{3} \leq i \leq \frac{2n-1}{3}, n = 8$ and 11
* $r(w_i'|\Pi) = (i - \frac{n-2}{3}, 0, i+2, \frac{2n+2}{3} - i)$, for $\frac{n+1}{3} \leq i \leq \frac{n+3}{2}, n \geq 17$ odd
* $r(w_i'|\Pi) = (i - \frac{n-2}{3}, 0, n-i+5, \frac{2n+2}{3} - i)$, for $\frac{n+5}{2} \leq i \leq \frac{2n-1}{3}, n \geq 17$ odd
* $r(w_i'|\Pi) = (i - \frac{n-2}{3}, 0, i+2, \frac{2n+2}{3} - i)$, for $\frac{n+1}{3} \leq i \leq \frac{n+2}{2}, n \geq 14$ even
* $r(w_i'|\Pi) = (i - \frac{n-2}{3}, 0, n-i+5, \frac{2n+2}{3} - i)$, for $\frac{n+4}{2} \leq i \leq \frac{2n-1}{3}, n \geq 14$ even
* $r(w_i'|\Pi) = (n-i+1, i - \frac{2n-4}{3}, n-i+5, 0)$, for $\frac{2n+2}{3} \leq i \leq \frac{3n-3}{3}$
* $r(w_n'|\Pi) = (0, \frac{n-2}{3} + 2, 5, 1)$.

Since all vertices of Bo_n for $n \geq 3$ have distinct representations, Π is a resolving partition. Hence, $pd(Bo_n) \leq 4$. Therefore, $pd(Bo_n) = 4$. $\square$

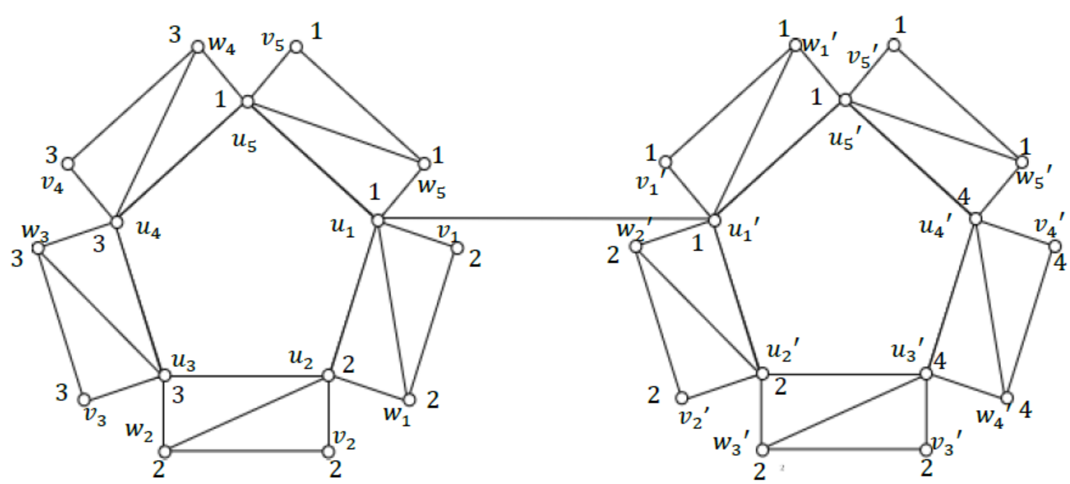

Figure 2. A minimum partition dimension of Bo_5.

3. Conclusion

The partition dimension of the origami graphs, $pd(O_n)$ is 3, whereas for barbell origami graphs, $pd(Bo_n)$ is 4 for $n \geq 3$.

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