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On interior Roman domination in graphs

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Abstract

Let $G = (V(G), E(G))$ be a non-complete graph and let $\phi : V(G) \to \{0, 1, 2\}$ be a function on G. For each $i \in \{0, 1, 2\}$, let $V_i = \{w \in V(G) : \phi(w) = i\}$. A function $\phi = (V_0, V_1, V_2)$ is an interior Roman dominating function (InRDF) on G if (i) for every $v \in V_0$, there exists $u \in V_2$ such that $uv \in E(G)$, and (ii) either $V_1 = V(G)$ or for every $z \in V_2$, z is an interior vertex of G. Denoted by $\omega_G^{InR}(\phi) = \sum_{u \in V(G)} \phi(u)$ is the *weight* of InRDF ϕ; and the minimum weight of an InRDF ϕ on G, denoted by $\gamma_{InR}(G)$, is called the *interior Roman domination number*. Any InRDF ϕ on graph G with $\omega_G^{InR}(\phi) = \gamma_{InR}(G)$ is called a γ_{InR}-function on G. In this paper, we introduce a new parameter of a Roman dominating function in graphs and discuss some important combinatorial properties.

Keywords: interior domination, Roman dominating function, interior Roman dominating function
Mathematics Subject Classification : 05C69

1. Introduction

Domination in graphs is one of the interesting topics in graph theory. Apparently, there are various domination parameters that have already been published in the literature, and it has a useful application in networking and protection strategies [9]. Nowadays, the theory of domination has been a center of motivation for many mathematicians to make research contributions to the field that involve theoretical properties [1], [2], [3], [4]. In the year 2004, Cockayne et al. [8] pioneered the foundation of Roman dominating function in graphs. Roman domination is directly inspired by the defense strategy in the fourth century A.D. carried out by the Roman Emperor Constantine the Great [8]. Since then, Roman domination has been a center of discrete mathematical research,

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and there are now several papers on different parameters that are published in the literature [7], [10] [12]. In 2016, the interior dominating set in graphs was initiated by Kinsley and Selvaraj [11]. The content of their paper introduced a new parameter of domination in graphs called interior domination and determined the bounds and their exact values in some classes of graphs. Inspired by the concepts of interior domination and Roman domination, the researcher combined the two concepts and initiated a new parameter called interior Roman domination in graphs.

As for the terminologies used in this study, the readers may refer to [5], [6], [9]. Let $G = (V(G), E(G))$ be any graph where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. The *order* of G denoted by $|V(G)|$ is defined as the cardinality of $V(G)$, and the *size* of G denoted by $|E(G)|$ is defined as the cardinality of $E(G)$. Let $v \in V(G)$. The *open neighborhood* of v in G is defined as the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of $u \in V(G)$ is defined as the set $N_G[u] = N_G(u) \cup \{u\}$. Let $O \subseteq V(G)$. The set $N_G(O) = N(O) = \bigcup_{u \in O} N_G(u)$ is called the *open neighborhood* of O and the set $N_G[O] = N[O] = N(O) \cup O$ is called the *closed neighborhood* of O in G. Let x and y be two distinct vertices in G. The *distance* between x and y is the length of the shortest path between x and y, and is denoted by $d_G(x, y)$. Now, if there is no such path from x to y, then we can define the distance between them as $d_G(x, y) = \infty$. Let $z \in V(G)$. If $d_G(x, y) = d_G(x, z) + d_G(z, y)$, then z is said to lie between x and y. In that case, z is called *interior vertex* of G.

For concepts and definitions of some classes of graphs that are not presented here, the readers may look into the following references: [2], [5], [6], [9]. A subset D of $V(G)$ is a *dominating set* of G if for each $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N[D] = V(G)$ [9]. The *domination number* of G is defined as the minimum cardinality of a dominating set $D \subseteq V(G)$ and is denoted by $\gamma(G)$. Moreover, D is called γ-set in G if $|D| = \gamma(G)$. Let $I \subseteq V(G)$ for which every element in I is an interior vertex in G. Then I is called interior dominating set of graph G provided that for every $v \in V(G) \setminus I$ there exists $u \in I$ such that $uv \in E(G)$, that is, $V(G) = N_G[I]$ [11]. The *interior domination number* of G is defined as the smallest cardinality of I in G and is denoted by $\gamma_{In}(G)$. In addition, a set I is called γ_{In}-set in G if $|I| = \gamma_{In}(G)$.

Let $\phi : V(G) \to \{0, 1, 2\}$ be a function on graph G and for each $j \in \{0, 1, 2\}$, let $V_j = \{w \in V(G) : \phi(w) = j\}$. Then we can write ϕ as $\phi = (V_0, V_1, V_2)$. In that case, a function $\phi : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if for each $v \in V_0$ there exists $u \in V_2$ such that $uv \in E(G)$ [8]. The *weight* of an RDF ϕ denoted by $\omega_G^R(\phi)$ is defined by $\omega_G^R(\phi) = \sum_{x \in V(G)} \phi(x) = |V_1| + 2|V_2|$. The *Roman domination number* of G is defined as the minimum weight of an RDF on G, that is, $\gamma_R(G) = min\{\omega_G^R(f) : \phi \textit{ is an } \text{RDF} \textit{ on } G\}$ and is denoted by $\gamma_R(G)$. On the face of it, every RDF ϕ on G with $\omega_G^R(\phi) = \gamma_R(G)$ is called a γ_R-function on G. Motivated by the concept of interior and Roman dominations, the following definition is realized.

Let G be a graph. Then a function $\phi = (V_0, V_1, V_2)$ is an interior Roman dominating function (InRDF) on G if and only if the following hold:

1. for every $v \in V_0$, there exists $u \in V_2$ such that $uv \in E(G)$; and
2. either $V_1 = V(G)$, or for every $z \in V_2$, z is an interior vertex of G.

The *weight* of InRDF ϕ is denoted by $\omega_G^{InR}(\phi)$ is the sum $\omega_G^{InR}(\phi) = \sum_{u\in V(G)} \phi(u)$; and the minimum weight of an InRDF on G, denoted by $\gamma_{InR}(G)$, is called *interior Roman domination number*. Consequently, an InRDF ϕ with weight $\gamma_{InR}(G)$ is regarded as a γ_{InR}-function on G.

In this paper, a new variant of Roman domination in graphs was introduced, called interior Roman domination. The theoretical properties of an interior Roman dominating function in any graph were explored. Furthermore, characterizations were constructed, lower and upper bounds of the interior Roman domination number were given, and a realization problem was discussed.

2. Results

This section explores some properties of the interior Roman dominating function on any simple and undirected graph.

Proposition 2.1. *Let G be a non-complete graph. If $\phi = (V_0, V_1, V_2)$ is an InRDF on G for which $V_1 = \varnothing$, then $V_2 \neq \varnothing$ is an interior dominating set on G.*

Proof. Suppose that $\phi = (V_0, V_1, V_2)$ is an InRDF on G. Then for every $v \in V_0$, there exists $u \in V_2$ such that u and v are adjacent in G, and either $V_1 = V(G)$ or for every $w \in V_2$, w is an interior vertex in G. Hence, if $V_1 = \varnothing$, then it follows that V_2 is a dominating set, that is, $V(G) = N_G[V_2]$. Therefore, it suffices to conclude that $V_2 \neq \varnothing$ is an interior dominating set on G. This completes the proof. □

In the case of Proposition 2.1, the set V_2 can be empty for some non-complete graph. For instance, let $G = \overline{K_n}$. Since there exists no interior vertex in G, it follows that $\gamma_{InR}(G) = |V_1| + 2|V_2| = |V_1| + 2(0) = |V_1| = |V(G)| = n$. Moreover, in general, Proposition 2.1 implies that the set V_2 is an interior dominating set on the graph $\langle V(G) \setminus V_1\rangle$. By definition of InRDF on any graph, the following remark is immediate.

Remark 2.1. Let G be a graph. If $\phi = (V_0, V_1, V_2)$ is an InRDF on G for which $|V_0| = |V_2| \geq 0$, then $\gamma_{InR}(G) = n$.

Proposition 2.2. *Let G be any graph of order n and let $\phi = (V_0, V_1, V_2)$ be γ_{InR}-function on G. Then $\gamma_{InR}(G) < n$ if and only if $|V_2| < |V_0|$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{InR}-function on any graph G with $|V(G)| = n$.
Assume that $\gamma_{InR}(G) < n$. By Remark 2.1, we get $|V_0| \neq |V_2|$. Suppose that if $|V_2| > |V_0|$, then we have

$$\begin{aligned}\gamma_{InR}(G) &= |V_1| + 2|V_2| \\ &> |V_1| + |V_2| + |V_0| \\ &= |V(G)| \\ &= n.\end{aligned}$$

A contradiction to our assupmtion that $\gamma_{InR}(G) < n$. Therefore, we obtain $|V_2| < |V_0|$. Conversely, we assume that $|V_2| < |V_0|$. Then it follows that

$$\begin{aligned}\gamma_{InR}(G) &= |V_1| + 2|V_2| \\ &< |V_1| + |V_2| + |V_0| \\ &= |V(G)| \\ &= n.\end{aligned}$$

This completes the proof. □

Proposition 2.3. *Let G be a graph and let $\phi = (V_0, V_1, V_2)$ be an InRDF on G with $V_1 = \varnothing$. Then V_2 is a γ_{In}-set on G if and only if $\phi = (V_0, V_1, V_2)$ is γ_{InR}-function on G.*

Proof. Suppose that $\phi = (V_0, V_1, V_2)$ InRDF on any graph G for which $V_1 = \varnothing$ and V_2 is a γ_{In}-set on G. Seeking for contradiction. Assume for a moment that $\phi = (V_0, V_1, V_2)$ is not a γ_{InR}-function on G. Then there exists γ_{InR}-function $\beta = (U_0, U_1, U_2)$ on G such that $U_1 = \varnothing$ and we have

$$\begin{aligned}\gamma_{InR}(G) = \omega_G^{InR}(\beta) &= |U_1| + 2|U_2| \\ &= 2|U_2| \\ &< 2|V_2| \\ &= |V_1| + 2|V_2| = \omega_G^{InR}(\phi).\end{aligned}$$

Thus, we obtain $|U_2| < |V_2|$ and, since U_2 is an interior dominating set by Proposition 2.1, this is a contradiction to our assumption that V_2 is a γ_{In}-set on G. Therefore, it suffices to conclude that $\phi = (V_0, V_1, V_2)$ is a γ_{InR}-function on G. Conversely, suppose $\phi = (V_0, V_1, V_2)$ is a γ_{InR}-function on G. Seeking for contradiction. Assume for a moment that V_2 is not a γ_{In}-set on G. Then there exists γ_{In}-set V_2' on G such that for every $x \in V_2'$, x is an interior vertex on G. Let $W_0 = V(G) \setminus V_2'$, $W_1 = \varnothing$, and $W_2 = V_2'$. Then is easy to check that $f' = (W_0, W_1, W_2)$ is an InRDF on G. Observe that $\omega_G^{InR}(f') = |W_1| + 2|W_2| = 2|V_2'| < 2|V_2| = |V_1| + 2|V_2| = \omega_G^{InR}(\phi) = \gamma_{InR}(G)$. Thus, we get $\omega_G^{InR}(f') < \gamma_{InR}(G)$. A contradiction to our assumption. Therefore, it suffices to say that V_2 is a γ_{In}-set on G. This completes the proof. □

The following results are values of interior Roman domination number for some special graphs.

Proposition 2.4. *Let n be a positive integer. Then $\gamma_{InR}(K_n) = n$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{InR}-function on K_n. Suppose that $n = 1, 2$. Then it is easy to check that $\gamma_{InR}(K_n) = |V_1| = |V(K_n)| = n$. Now, suppose that $n \geq 3$. Since for arbitrary distinct vertices $u, v, w \in V(K_n)$, $d_{K_n}(u, w) \neq d_{K_n}(u, v) + d_{K_n}(v, w)$, it follows that there is no interior vertex in G. Consequently, $V_2 = \varnothing$ and so, $V_0 = \varnothing$. Thus, we get $\gamma_{InR}(G) = |V_1| + 2|V_2| = |V_1| + 2(0) = |V(G)| = n$. This completes the proof. □

Proposition 2.5. *For an integer $n \geq 3$, we have*

$$\gamma_{InR}(P_n) = \gamma_{InR}(C_n) = \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \pmod 3 \\ \frac{2n+1}{3}, & \text{if } n \equiv 1 \pmod 3 \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Proof. Let $P_n = [v_1, v_2, ..., v_n]$ and $C_n = [u_1, u_2, ..., u_n, u_1]$ where $n \geq 3$. In addition, let $\phi_{P_n} = (V_0^{P_n}, V_1^{P_n}, V_2^{P_n})$ be a γ_{InR}-function on P_n and $\phi_{C_n} = (V_0^{C_n}, V_1^{C_n}, V_2^{C_n})$ be a γ_{InR}-function on C_n. Then consider the following cases:

Case 1. $n \equiv 0 \pmod 3$
Set $V_2^{P_n} = \{v_2, v_5, v_8, ..., v_{n-1}\}$ and $V_2^{C_n} = \{u_2, u_5, u_8, ..., u_{n-1}\}$. Then, we obtain $V_0^{P_n} = V(P_n) \setminus V_2^{P_n}$, $V_0^{C_n} = V(C_n) \setminus V_2^{C_n}$, $V_1^{P_n} = \varnothing$ and $V_1^{C_n} = \varnothing$. In that case, $V(P_n) = N_{P_n}[V_2^{P_n}]$ and $V(C_n) = N_{C_n}[V_2^{C_n}]$. Moreover, it is clear that $V_2^{P_n}$ and $V_2^{C_n}$ are minimum interior dominating sets on P_n and C_n, respectively. Thus, by Proposition 2.3, it follows that $\gamma_{InR}(P_n) = |V_1^{P_n}| + 2|V_2^{P_n}| = 0 + 2\left(\frac{n}{3}\right) = \frac{2n}{3}$ and $\gamma_{InR}(C_n) = |V_1^{C_n}| + 2|V_2^{C_n}| = 0 + 2\left(\frac{n}{3}\right) = \frac{2n}{3}$.

Case 2. $n \equiv 1 \pmod 3$
For P_n, we set $V_2^{P_n} = \{v_2, v_5, v_8, ..., v_{n-2}\}$ and $V_1^{P_n} = \{v_n\}$, and for C_n, we set $V_2^{C_n} = \{u_2, u_5, u_8, ..., u_{n-2}\}$ and $V_1^{C_n} = \{u_n\}$. Then, we get $V_0^{P_n} = V(P_n) \setminus \left(V_1^{P_n} \cup V_2^{P_n}\right)$ and $V_0^{C_n} = V(C_n) \setminus \left(V_1^{C_n} \cup V_2^{C_n}\right)$. So, it implies that $V(P_n) = N_{P_n}[V_1^{P_n} \cup V_2^{P_n}]$ and $V(C_n) = N_{C_n}[V_1^{C_n} \cup V_2^{C_n}]$. Now, it is easy to check that $V_2^{P_n}$ and $V_2^{C_n}$ are minimum interior dominating sets on P_n and C_n, respectively. Consequently, if we invoke Proposition 2.3, then it implies that $\gamma_{InR}(P_n) = |V_1^{P_n}| + 2|V_2^{P_n}| = 1 + 2\left(\frac{n-1}{3}\right) = \frac{2n+1}{3}$ and $\gamma_{InR}(C_n) = |V_1^{C_n}| + 2|V_2^{C_n}| = 1 + 2\left(\frac{n-1}{3}\right) = \frac{2n+1}{3}$.

Case 3. $n \equiv 2 \pmod 3$
Set $V_2^{P_n} = \{v_2, v_5, v_8, ..., v_{n-1}\}$ and $V_2^{C_n} = \{u_2, u_5, u_8, ..., u_{n-1}\}$. This means that $V_0^{P_n} = V(P_n) \setminus V_2^{P_n}$, $V_0^{C_n} = V(C_n) \setminus V_2^{C_n}$, $V_1^{P_n} = \varnothing$, and $V_1^{C_n} = \varnothing$. In that case, we obtain $V(P_n) = N_{P_n}[V_2^{P_n}]$ and $V(C_n) = N_{C_n}[V_2^{C_n}]$. Clearly, it follows that $V_2^{P_n}$ and $V_2^{C_n}$ are minimum interior dominating sets on P_n and C_n, respectively. Thus, in view of Proposition 2.3, we end up with $\gamma_{InR}(P_n) = |V_1^{P_n}| + 2|V_2^{P_n}| = 0 + 2\left(\frac{n+1}{3}\right) = \frac{2n+2}{3}$ and $\gamma_{InR}(C_n) = |V_1^{C_n}| + 2|V_2^{C_n}| = 0 + 2\left(\frac{n+1}{3}\right) = \frac{2n+2}{3}$. This completes the proof. □

Proposition 2.6. *Let G be a graph. Then $\gamma_{InR}(G) = 1$ if and only if $G = K_1$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{InR}-function on G. Suppose that $\gamma_{InR}(G) = 1$. Then it follows that $|V_1| + 2|V_2| = 1$ and hence, $|V_1| = 1$ and $|V_2| = 0$. Moreover, it implies that $V_0 = \varnothing$. Thus, we have $|V(G)| = |V_0| + |V_1| + |V_2| = 0 + 1 + 0 = 1$ and so, $G = K_1$. The converse is clear. This completes the proof. □

Theorem 2.1. *Let G_i $(i = 1, ..., k)$ be components of graph G. Then ϕ is an InRDF on G if and only if $\phi|_{G_i}$ is an InRDF on G_i for all $i \in \{1, 2, ...k\}$. In particular,*

$$\gamma_{InR}(G) = \sum_{i=1}^{k} \gamma_{InR}(G_i).$$

Proof. Suppose that $G_1, ..., G_k$ are the components of graph G with $|V(G)| = n$ and let $\phi = (V_0, V_1, V_2)$ be an RDF on G. Assume that ϕ is an InRDF on G. For each $i \in \{1, 2, ..., k\}$, we set $V_0^i = V_0 \cap V(G_i)$, $V_1^i = V_1 \cap V(G_i)$ and $V_2^i = V_2 \cap V(G_i)$. So we obtain $\phi|_{G_i} = (V_0^i, V_1^i, V_2^i)$ for each $i \in \{1, 2, ..., k\}$. Let $v \in V_0^i$ for some $i \in \{1, 2, ..., k\}$. Since ϕ is an InRDF on G, it follows that there exists $u \in V_2$ such that $uv \in E(G)$ and u is an interior vertex of G. Since for any $x \in V(G_k)$ and for any $y \in V(G_l)$ $(k \neq l)$, $d_G(x, y) = \infty$, it implies that $u \in V_2^i$ and

$d_{G_j}(u,v)=1$ where u is an interior vertex in G_i, showing that $\phi|_{G_i}$ is an InRDF on G_i. Therefore, it suffices to say that $\phi|_{G_i}$ is an InRDF on G_i for all $i \in \{1,2,...,k\}$. Thus, if ϕ is a γ_{InR}-function on G, then

$$\begin{aligned}\gamma_{InR}(G) &= |V_1| + 2|V_2| \\ &= \sum_{i=1}^{k} |V_1^i| + 2\sum_{i=1}^{k} |V_2^i| \\ &= \sum_{i=1}^{k} \left(|V_1^i| + 2|V_2^i|\right) \\ &\geq \sum_{i=1}^{k} \gamma_{InR}(G_i).\end{aligned}$$

Assume that $\phi|_{G_i} = (U_0^i, U_1^i, U_2^i)$ is an InRDF on G_i for all $i \in \{1,2,...k\}$. Let $V_0 = \bigcup_{i=1}^{k} U_0^i$, $V_1 = \bigcup_{i=1}^{k} U_1^i$ and $V_2 = \bigcup_{i=1}^{k} U_2^i$. Then $\phi = (V_0, V_1, V_2)$ is an RDF on G. Now, let $v' \in V_0$. Then we have $v' \in V_0^i$ for some $i \in \{1,2,...k\}$. Since $\phi|_{G_i}$ is an InRDF on G, it implies that there exists $u' \in V_2^i$ such that $u'v' \in E(G_i)$ where u' is an interior vertex on G_i. Since $V_2^i \subseteq V_2$, it follows that $u' \in V_2$, showing that ϕ is an InRDF G. Thus, if $\phi|_{G_i}$ is a γ_{InR}-function on G_i for all $i \in \{1,2,...k\}$, then

$$\begin{aligned}\sum_{i=1}^{k} \gamma_{InR}(G_i) &= \sum_{i=1}^{k} \left(|V_1^i| + 2|V_2^i|\right) \\ &= \sum_{i=1}^{k} |V_1^i| + 2\sum_{i=1}^{k} |V_2^i| \\ &= |V_1| + 2|V_2| \\ &\geq \gamma_{InR}(G).\end{aligned}$$

In particular, we obtain

$$\gamma_{InR}(G) = \sum_{i=1}^{k} \gamma_{InR}(G_i).$$

This completes the proof. □

The next theorem is a realization problem.

Theorem 2.2. *Let a, b, and n be positive integers for which $2 \leq a \leq b \leq n$. Then there exists a graph G of order n such that $\gamma_R(G) = a$ and $\gamma_{InR}(G) = b$.*

Proof. Let G be any graph and let $\lambda = (U_0, U_1, U_2)$ be a γ_R-function on G and let $\phi = (V_0, V_1, V_2)$ be a γ_{InR}-function on G. Then consider the following cases:
Case 1. $2 \leq a \leq b = n$
If $G = K_2$, then it is clear that $\gamma_R(G) = \gamma_{InR}(G) = 2$. Now, let $G = K_n$ where $n \geq 3$. Then let $u \in V(G)$ and set $U_0 = V(G) \setminus \{u\}$, $U_1 = \varnothing$ and $U_2 = \{u\}$. This implies that $\gamma_R(G) = \omega_G^R(\lambda) = |U_1| + 2|U_2| = 0 + 2(1) = 2$. On the other hand, by Proposition 2.4, we obtain $\gamma_{InR}(G) = \omega_G^{InR}(\phi) = |V_1| = |V(G)| = n$. Since $n \geq 3$, it follows that $2 = \gamma_R(G) < \gamma_{InR}(G) = n$. Thus, the conclusion holds.
Case 2. $2 < a < b < n$
Let $G = \big(\cup_{i=1}^k K_m^i \big) \cup P_3$ where $k \geq 1$, $m \geq 3$ and K_m^i is a complete graph of order m for each $i \in \{1, 2, ..., k\}$. It follows that $|V(G)| = n = km + 3$ where $k \geq 1$ and $m \geq 3$. Let $u_i \in V(K_m^i)$ for all $i \in \{1, 2, ..., k\}$. Then, we have $V\big(\cup_{i=1}^k K_m^i \big) = N[\cup_{i=1}^k \{u_i\}]$. Also, we let $v \in V(P_3)$ for which $V(P_3) = N_{P_3}[v]$. In that case, we set $U_0 = V(G) \setminus \big(\big(\cup_{i=1}^k \{u_i\} \big) \cup \{v\} \big)$, $U_1 = \varnothing$, and $U_2 = \big(\cup_{i=1}^k \{u_i\} \big) \cup \{v\}$. So, we have $\gamma_R(G) = \omega_G^R(\lambda) = |U_1| + 2|U_2| = 0 + 2(k+1) = 2k + 2$. Note that if $v \in V(P_3)$ and $V(P_3) = N_{P_3}[v]$, then $deg_G(v) = 2$ and so, it is clear that v is an interior vertex in P_3. Thus, we have $\gamma_{InR}(P_3) = 2$. In view of Proposition 2.4, we have $\gamma_{InR}\big(\cup_{i=1}^k K_m^i \big) = |V_1| = \Big| V\big(\cup_{i=1}^k K_m^i \big) \Big| = km$. Invoking Theorem 2.1, it implies that $\gamma_{InR}(G) = \gamma_{InR}\big(\cup_{i=1}^k K_m^i \big) + \gamma_{InR}(P_3) = km + 2$. Since $k \geq 1$ and $m \geq 3$, it follows that $2 < 2k + 2 < km + 2 < n = km + 3$. So, the conclusion follows.
Case 3. $2 < a = b < n$
Consider $G = P_n$ where $6 \leq n \equiv 0 \pmod 3$. By Proposition 2.5, it follows that $2 < \gamma_R(G) = \frac{2n}{3} = \gamma_{InR}(G) < n$ and the conclusion follows.
Combining the three cases, completes the proof. □

As a consequence of Theorem 2.2, the following corollary is obtained.

Corollary 2.1. *Let G be a graph of order n. Then the difference $\gamma_{InR}(G) - \gamma_R(G)$ can be made arbitrarily large.*

Proof. Let m and n be positive integers such that $m + 2 < n$. In view of Theorem 2.2, there exists a graph G with $|V(G)| = n$ such that $\gamma_{InR}(G) = m + 2$ and $\gamma_R(G) = 2$. Then we obtain $\gamma_{InR}(G) - \gamma_R(G) = m$. As we increase m sufficiently large as possible, it follows that $\gamma_{InR}(G) - \gamma_R(G)$ can be made arbitrarily large. This completes the proof. □

Theorem 2.3. *Let $\phi = (V_0, V_1, V_2)$ be a γ_{InR}-function on a non-complete and connected graph G of order n for which $|V_1| = 0$. Then $\gamma_{InR}(G) = \gamma_{In}(G) + 1$ if and only if there is a vertex $v \in V(G)$ such that $deg_G(v) = n - \gamma_{In}(G)$.*

Proof. Let G be a non-complete and connected graph of order n and let $\phi = (V_0, V_1, V_2)$ be a γ_{InR}-function on G for which $|V_1| = 0$.
Suppose that $\gamma_{InR}(G) = \gamma_{In}(G) + 1$. Then it follows that $\gamma_{In}(G) + 1 = |V_1| + 2|V_2|$. Since $V_1 = \varnothing$, by Proposition 2.1, it follows that V_2 is an interior dominating set on G. And since ϕ be a γ_{InR}-function on G for which $|V_1| = 0$, by Proposition 2.3, it implies that V_2 is a γ_{In}-set on G.

Thus, we have $|V_2|+1 = 2|V_2|$ and so, $|V_2| = 1$. Let $V_2 = \{v\}$. Since V_2 is a γ_{In}-set on G, we have $V(G) = N_G[v]$. Note that G is non-complete and connected graph with $|V(G)| = n$. Therefore, we end up with $deg_G(v) = N_G(v) = |V(G)| - |\{v\}| = n - |V_2| = n - \gamma_{In}(G)$.

Suppose that there exists a vertex $v \in V(G)$ such that $deg_G(v) = n - \gamma_{In}(G)$. Since ϕ be a γ_{InR}-function on G for which $|V_1| = 0$, it implies that V_2 is a γ_{In}-set on G, that is, $|V_2| = \gamma_{In}(G)$. So, we get $deg_G(v) = N_G(v) = |V(G)| - |V_2| = |V_0| + |V_1| + |V_2| - |V_2| = |V_0|$. Thus, it follows that $V(G) = N_G[v]$. Since G is a non-complete and connected graph, then there exists $x, y \in V(G)$ distinct from v such that $d_G(x, y) = d_G(x, v) + d_G(v, y)$. This implies that v is an interior vertex of G. Let $V_2 = \{v\}$. Then $|V_2| = 1$. Therefore, we obtain $\gamma_{InR}(G) = \omega_G^{InR}(\phi) = |V_1| + 2|V_2| = |V_2| + |V_2| = \gamma_{In}(G) + 1$. □

The following result portrays the lower and upper bounds of interior Roman domination number of any graph G.

Theorem 2.4. *Let G be a graph with $|V(G)| = n$ and $\phi = (V_0, V_1, V_2)$ be a γ_{InR}-function on G for which $V_1 = \varnothing$. Then,*

$$max\{\gamma_R(G), \gamma_{In}(G)\} \leq \gamma_{InR}(G) \leq min\{2\gamma_{In}(G), n\}.$$

Proof. Assume that $\phi = (V_0, V_1, V_2)$ is a γ_{InR}-function on any graph G of order n such that $V_1 = \varnothing$. Invoking Proposition 2.1, it implies that V_2 is an interior dominating set on G. So, we have $\gamma_{In}(G) \leq |V_2| \leq |V_1| + 2|V_2| = \omega_G^{InR}(\phi) = \gamma_{InR}(G)$. Thus, $\gamma_{In}(G) \leq \gamma_{InR}(G)$. Since every interior Roman dominating function is a Roman dominating function on G, it follows that $\gamma_R(G) \leq \gamma_{InR}(G)$. Therefore, the lower bound of $\gamma_{InR}(G)$ is equal to $max\{\gamma_{In}(G), \gamma_R(G)\}$. Meanwhile, if we set $V_0 = \varnothing$, then we get $V_2 = \varnothing$. In that case, $\phi = (\varnothing, V_1 = V(G), \varnothing)$ is an interior Roman dominating function on G. This implies that $\gamma_{InR}(G) \leq \omega_G^{InR}(\phi) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. Let V_2 be a γ_{In}-set on G. Suppose that $\phi = (V_0, V_1, V_2)$ is a γ_{InR}-function on G such that $V_1 = \varnothing$. Since V_2 is an interior dominating set on G, it follows that $\gamma_{InR}(G) \leq \omega_G^{InR}(\phi) = |V_1| + 2|V_2| = 2|V_2| = 2\gamma_{In}(G)$. Thus, the upper bound of $\gamma_{InR}(G)$ is equal to $min\{n, 2\gamma_{InR}(G)\}$. This completes the proof. □

The bounds in Theorem 2.4 are sharp. To see this, let $G = K_n$ for all $n \geq 1$. Then it follows that $\gamma_{In}(G) = \gamma_{InR}(G) = |V_1| = n$. Let $G = K_1 + P_n$ for all $n \geq 3$. Since $v \in V(K_1)$ is an interior vertex in G, it follows that $\gamma_R(G) = \gamma_{InR}(G) = 2$. Finally, let $G = P_4 = [v_1, v_2, v_3, v_4]$ and let $\phi = (V_0, V_1, V_2)$ be γ_{InR}-function on G. In that case, we set $V_1 = \varnothing$ and $V_2 = \{v_2, v_3\}$ where v_2 and v_3 are interior vertices in P_4. This follows that $\gamma_{In}(G) = |V_2|$. Hence, we obtain $\gamma_{InR}(G) = \omega_G^{InR}(\phi) = |V_1| + 2|V_2| = 2|V_2| = 2\gamma_{In}(G)$.

Let G and H be any graphs. The *join* of graphs G and H is the graph $G + H$ with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G)$ and $v \in V(H)\}$ and is denoted as $G + H$.

Theorem 2.5. *Let G be a non-complete connected graph and H be a non-complete graph. Then $\gamma_{InR}(G) = 2$ if and only if $G = K_1 + H$ where $|V(H)| \geq 2$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a γ_{InR}-function on a non-complete connected graph G. Then it follows that for any $v \in V_2$, v is an interior vertex in G and if $V_1 = \varnothing$, by Proposition 2.1, V_2 is an interior dominating set on G.

Assume that $\gamma_{InR}(G) = 2$. Then it follows that $|V_1| + 2|V_2| = 2$ and so, $|V_1| = 2$ or $|V_2| = 1$. Suppose that $|V_1| = 2$. Then, it implies that $V_2 = \varnothing$ and so, $|V_0| = 0$. Hence, $\gamma_{InR}(G) = \omega_G^{InR}(\phi) = |V_1| = |V(G)| = 2$ and thus, $G \in \{K_2, \overline{K_2}\}$. This contradicts to our assumption that G is a non-complete connected graph. Suppose that $|V_2| = 1$. Then, we get $V_1 = \varnothing$. Now, let $V_2 = \{u\}$ where u is an interior vertex in G. Since V_2 is an interior dominating set on G, it follows that there exists two distinct vertices $x, y \in V(G) \setminus V_2$ such that $d_G(x, y) = d_G(x, u) + d_G(u, y)$ and $V(G) = N_G[V_2]$. Hence, it suffices to conclude that $G = K_1 + H$, where $u \in V(K_1)$ and $x, y \in H$. Therefore, H is a non-complete graph with $|V(H)| \geq 2$.

Assume that $G = K_1 + H$ where H is a non-complete graph with $|V(H)| \geq 2$. Let $u \in V(K_1)$. Since H is a non-complete graph with $|V(H)| \geq 2$, it implies that there exists two distinct vertices, say, x and y, such that $d_G(x, y) = 2$. So, it follows that $d_G(x, y) = d_G(x, u) + d_G(u, y)$ and thus, u is an interior vertex in G. Moreover, we get $V(G) = N_G[u]$ and hence, $\{u\}$ is minimum interior dominating set on G. In that case, we set $V_2 = \{u\}$, $V_1 = \varnothing$, and $V_0 = V(G) \setminus V_2 = V(H)$. Therefore, we end up with $\gamma_{InR}(G) = |V_1| + 2|V_2| = 0 + 2(1) = 2$.
This completes the proof. □

The following corollary is immediate from Theorem 2.5.

Corollary 2.2. *Let n be a positive integer. Then the following holds:*

i.) $\gamma_{InR}(S_n) = 2$ *where* $n \geq 2$*;*
ii.) $\gamma_{InR}(W_n) = 2$ *where* $n \geq 4$*;*
iii.) $\gamma_{InR}(F_n) = 2$ *where* $n \geq 3$*; and*
iv.) $\gamma_{InR}(K_{m,n}) = 4$ *where* $m, n \geq 2$*;*

The next theorem is a characterization of interior Roman dominating function in the join of two graphs.

Theorem 2.6. *Let G and H be non-complete connected graphs. Then $\phi = (V_0, V_1, V_2)$ is a InRDF on $G + H$ if and only if one of the following is satisfied:*

i.) there exist $x_1, x_2 \in V_2$ such that $x_1 \in V(G)$ and $x_2 \in V(H)$ are interior vertices; or
ii.) $V_1 \cup V_2 = V_1^G \cup V_2^G$*; or*
iii.) $V_1 \cup V_2 = V_1^H \cup V_2^H$*; or*
iv.) $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$.

where $\phi|_G = (V_0^G, V_1^G, V_2^G)$ and $\phi|_H = (V_0^H, V_1^H, V_2^H)$ are InRDF on G and H, respectively, such that for each $i \in \{0, 1, 2\}$, $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$.

Proof. Let $\phi = (V_0, V_1, V_2)$ be an RDF on $G + H$ where G and H are non-complete connected graphs. Assume that ϕ is an InRDF on $G + H$. Since G and H are non-complete connected graphs, it follows that there exist interior vertices x_1 and x_2 such that $x_1 \in V(G)$ and $x_2 \in V(H)$. Let $x_1, x_2 \in V_2$. Since $V(G) \subseteq N_{G+H}[x_2]$ and $V(H) \subseteq N_{G+H}[x_1]$, it implies that $V(G) = N_{G+H}[V_2]$.

Hence, $(i.)$ holds. On the other hand, set $I_G = (V_1 \cup V_2) \cap V(G)$ and $I_H = (V_1 \cup V_2) \cap V(H)$. Then let $v \in V(G) \setminus I_G \subseteq V_0$. Since ϕ is an InRDF on $G+H$, it follows that there exists $u \in V_2$ such that $uv \in E(G+H)$ and u is an interior vertex in $G+H$. Suppose that if $u \in I_G \subseteq V(G)$, then it means that $I_G \setminus V_1$ is an interior dominating set on the subgraph $\langle V(G) \setminus V_1 \rangle$. Now, let $V_0^G = V(G) \setminus I_G$, $V_1^G = I_G \setminus V_2$ and $V_2^G = I_G \setminus V_1$. Thus, we obtain $V_i^G = V_i \cap V(G)$ for each $i \in \{0, 1, 2\}$ and so, $\phi|_G = (V_0^G, V_1^G, V_2^G)$ is an InRDF on G. Also, let $V_0^H = V(H) \setminus I_H$, $V_1^H = I_H \setminus V_2$ and $V_2^H = I_H \setminus V_1$. Then by similar argument, it implies that $V_i^H = V_i \cap V(H)$ for any $i \in \{0, 1, 2\}$ and so, $\phi|_H = (V_0^H, V_1^H, V_2^H)$ is an InRDF on H. Now, since $V(G+H) = N_{G+H}[V_1^G \cup V_2^G]$, it follows that $V_1 \cup V_2 = V_1^G \cup V_2^G$ and so, $(ii.)$ holds. Also, since $V(G+H) = N_{G+H}[V_1^H \cup V_2^H]$, it also follows that $V_1 \cup V_2 = V_1^H \cup V_2^H$ and thus, $(iii.)$ holds. Accordingly, we also have $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$ and $(iv.)$ is also satisfied. Conversely, assume that $(i.)$ holds, that is, there exists $x_1, x_2 \in V_2$ such that $x_1 \in V(G)$ and $x_2 \in V(H)$ are interior vertices. Note that $V(G) \subseteq N_{G+H}[x_2]$ and $V(H) \subseteq N_{G+H}[x_1]$. Hence, $V(G+H) = N_{G+H}[\{x_1, x_2\}] = N_{G+H}[V_2]$. Consequently, $\phi = (V_0, V_1, V_2)$ is an InRDF on $G+H$. Now, assume that $(ii.)$ or $(iii.)$ or $(iv.)$ hold. Since $\phi|_G = (V_0^G, V_1^G, V_2^G)$ and $\phi|_H = (V_0^H, V_1^H, V_2^H)$ are InRDF on G and H, respectively, it clearly implies that $\phi = (V_0, V_1, V_2)$ is an InRDF on $G+H$.
This completes the proof. □

The corollaries below are immediate consequence from Theorem 2.6.

Corollary 2.3. *Let G and H be non-complete connected graphs. Then $2 \le \gamma_{InR}(G+H) \le 4$.*

Corollary 2.4. *Let G and H be non-complete connected graphs. Then*

$$\gamma_{InR}(G+H) = min\{4, \gamma_{InR}(G), \gamma_{InR}(H)\}.$$

The following remark is quick from Proposition 2.4 and Theorem 2.6.

Remark 2.2. Let G and H be complete graphs with $|V(G)| = n$ and $|V(H)| = m$. Then $\gamma_{InR}(G+H) = n+m$.

Let G and H be any graphs. The *corona* of G and H is defined to be the graph obtained by taking one copy of G and $|V(G)|$ copies of H and then forming the joins $\langle v \rangle + H^v = v + H^v$ for each $v \in V(G)$, where H^v is a copy of H corresponding to vertex v and is denoted by $G \circ H$. The following theorem is a characterization of interior Roman dominating function in the corona of two nontrivial connected graphs.

Theorem 2.7. *Let G and H be a nontrivial connected graphs and let $\phi = (V_0, V_1, V_2)$ be an RDF on $G \circ H$. Then ϕ is a γ_{InR}-function on $G \circ H$ if and only if $V_2 = V(G)$ and $V_1 = \varnothing$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be an RDF on $G \circ H$ where G and H are nontrivial connected graphs. Suppose that ϕ is a γ_{InR}-function on $G+H$. Seeking for contradiction. Assume for a moment that $V_2 \neq V(G)$ or $V_1 \neq \varnothing$. Consider $V_2 \neq V(G)$. Then it implies that $V_2 \subsetneq V(G)$ or $V(G) \subsetneq V_2$. On one hand, we suppose that $V_2 \subsetneq V(G)$. Then there exists $v \in V(G)$ such that $V(H^v) \nsubseteq N_{G \circ H}[V_2]$ and so, $V(G \circ H) \neq N_{G \circ H}[V_2]$. A contradiction. On the other hand, we suppose that $V(G) \subsetneq V_2$. Then it follows that there exists a vertex $u \in V_2 \setminus V(G)$ such that $u \in V(H^v)$ for some $v \in V(G)$. Now, if H is a complete graph, then by the proof of Proposition 2.4, u is not an interior vertex in $G \circ H$, a contradiction. Moreover, since $V(G \circ H) = N_{G \circ H}[V(G)]$, we can set $W_0 = V(G \circ H) \setminus V(G)$, $W_1 = \varnothing$ and $W_2 = V(G)$. In that case, $f = (W_0, W_1, W_2)$ is an InRDF on $G \circ H$. Observe that

$$\begin{aligned}
\omega_{G \circ H}^{InR}(f) &= |W_1| + 2|W_2| \\
&= 2|V(G)| \\
&< 2|V_2| \\
&= |V_1| + 2|V_2| \\
&= \gamma_{InR}(G \circ H).
\end{aligned}$$

A contradiction. Thus, we conclude that $V_2 = V(G)$. And since $V(G \circ H) = N_G[V_2]$ and ϕ is γ_{InR}-function on $G \circ H$, it suffices to conclude that $V_1 = \varnothing$.

Suppose that $V_2 = V(G)$ and $V_1 = \varnothing$. Let $v \in V_2 = V(G)$. Then it follows that there exists $x \in V(H^v)$ and $y \in V(G)$ such that $d_{G \circ H}(x, y) = d_{G \circ H}(x, v) + d_{G \circ H}(v, y)$. And so V_2 is an interior vertex set in $G \circ H$. Seeking for contradiction. Assume for a moment that $\phi = (V_0, V_1, V_2)$ is an InRDF but not a γ_{InR}-function on $G \circ H$. Then there exists $g = (U_0, U_1, U_2)$ such that g is a γ_{InR}-function on $G \circ H$ for which $U_1 = \varnothing$. It implies that $\gamma_{InR}(G \circ H) = |U_1| + 2|U_2| = 2|U_2| < 2|V_2| = |V_1| + 2|V_2| = \omega_{G \circ H}^{InR}(\phi)$. Thus, it means that $|U_2| < |V_2|$. Since $V_2 = V(G)$ and $V_1 = \varnothing$, it follows that there exists $V(H^v)$ for some $v \in V(G)$ such that $V(H^v) \nsubseteq N_{G \circ H}[U_2]$ and so, $V(G \circ H) \neq N_{G \circ H}[U_2]$. This is a contradiction. Therefore, it suffices to say that ϕ is a γ_{InR}-function on $G \circ H$. This completes the proof. □

The following corollary is immediate from Theorem 2.7.

Corollary 2.5. *Let G be a nontrivial connected graph with $|V(G)| = n$ and H be any nontrivial connected graph. Then $\gamma_{InR}(G \circ H) = 2n$.*

Conclusion

This study introduced a new variation of Roman domination in graphs called the interior Roman dominating function, and some important properties were presented. Additionally, it is depicted that for any graph G of order n, the lower bound of $\gamma_{InR}(G)$ is $max\{\gamma_R(G), \gamma_{In}(G)\}$ and the upper bound is $min\{2\gamma_{In}(G), n\}$. The study also portrayed that for positive integers a, b and n with $1 \leq a \leq b \leq n$, there exists a graph G such that $\gamma_R(G) = a$, $\gamma_{InR}(G) = b$ and $|V(G)| = n$. Moreover, this paper has shown that for any non-complete graph H, $\gamma_{InR}(G) = 2$ if and only if $G = K_1 + H$ where $|V(H)| \geq 2$. Furthermore, characterizations of the interior Roman dominating function in the join and corona of two graphs were discussed. Investigation on the characterization of interior Roman domination under the binary operations, such as Cartesian and lexicographic products, is highly recommended for future studies.

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