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Outer Multiset Dimension of Joined Graphs

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Abstract

The outer multiset dimension of a graph G, $dim_{ms}(G)$, is the cardinality of the smallest subset S of vertices that uniquely recognizes each vertex outside S by using the multiset of distances between the vertex and the vertices in S.

In 2023, Klavzar, Kuziak, and Yero proved that the only graphs with the largest outer multiset dimension, that is, one less than their order, are regular graphs of diameter at most 2. This paper considers the outer multiset dimensions of non-regular graphs of diameter 2 obtained from the join product, in particular, stars, wheels, generalized wheels, windmills, fans, and generalized fans.

Keywords: metric dimension, outer multiset dimension, join product, joined graph
Mathematics Subject Classification: 05C12, 05C76

1. Introduction

One of the classic topics in metric graph theory is the computation of the metric dimension of graphs. The concept of metric dimension applies to several location-related issues in fields such as social sciences, computer science, chemistry, and biology [7].

Slater [5] and Harary and Melter [3] independently proposed the concepts of resolving sets and metric dimension in 1975 and 1976, respectively. Here, we use the terms of Melter and Harary.

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Let $G = (V, E)$ be a simple connected graph of order $n(G)$ and diameter $diam(G)$. Given an ordered set of vertices $S = \{w_1, \ldots, w_t\} \subseteq V$, the *metric representation* of a vertex $u \in V$ with respect to S is the t-vectors $r(u|S) = (d_G(u, w_1), \ldots, d_G(u, w_t))$. If every two vertices u and w have different metric representations, i.e. $r(u|S) \neq r(w|S)$, then S is a *resolving set*. The *metric dimension* of G, $dim(G)$, is the cardinality of the smallest resolving set.

For $S = \{w_1, \ldots, w_t\} \subseteq V$, the *multiset representation* of $u \in V(G)$ with respect to S is defined as $m(u|S) = \{d_G(u, w_1), \ldots, d_G(u, w_1)\}$. A set S such that two vertices of G have different multiset representations is called a *multiset resolving set*. If G has a multiset resolving set, then the *multiset dimension* of G, $md(G)$, is the cardinality of the smallest multiset resolving set, otherwise, G has an infinite multiset dimension [1, 6].

The concept of an outer multiset resolving set was proposed to address the issue of infiniteness in the multiset metric dimension. A set S such that two vertices outside of S have different multiset representations is called an *outer multiset resolving set*. The cardinality of the smallest outer multiset resolving set is called the *outer multiset dimension* of G, denoted as $dim_{ms}(G)$ [2, 4]. Since only vertices outside of a set S must be distinguished, this approach effectively resolves the issue of infiniteness in the multiset dimension. It is also obvious that $dim_{ms}(G) \leq n(G) - 1$, and the characterization of graphs with the smallest and largest outer multiset dimension is given below.

Theorem 1.1. *[2] A graph G satisfies $dim_{ms}(G) = 1$ if and only if it is a path graph.*

Theorem 1.2. *[4] A graph G satisfies $dim_{ms}(G) = n(G) - 1$ if and only if G is regular with $diam(G) \leq 2$.*

This paper considers the outer multiset dimension of some joined graphs. Recall that the *join* of two graphs G and H, denoted as $G + H$, is constructed by taking two graphs and adding an edge between every vertex in G and every vertex in H. Thus, the vertex set of $G + H$ is the union of the set of vertices of G and H, and the edge set consists of the edges within G, the edges within H, and all possible edges connecting a vertex from G to a vertex from H. We also define the *complement* of a graph G, $\overline{G}$, as the graph with the same set of vertices but with the opposite set of edges: an edge exists in $\overline{G}$ if and only if it does not exist in G. We shall consider the following joined graphs.

- the stars $S_n = \overline{K_{n-1}} + K_1$,
- the wheels $W_n = C_{n-1} + K_1$,
- generalized wheels $W_{m,n} = C_n + \overline{K_m}$,
- the windmills $W(k, n) = nK_k + K_1$.
- the fans $F_n = P_{n-1} + K_1$, and
- generalized fans $F_{m,n} = P_n + \overline{K_m}$.

Note that all non-complete joined graphs have diameter 2. Moreover, the majority of the joined graphs under consideration are non-regular. By Theorem 1.2, we have the following for all graphs considered.

Corollary 1.1. *For integers n, m, k, if G is a graph in $\{S_n, F_n, F_{m,n}, W_n, W_{m,n}, W(k,n)\}$, then*

$$dim_{ms}(G) \leq n(G) - 2.$$

2. Main Results

Before presenting the results, we define the terminology and notation used throughout this section. For a positive integer k, we write $[k] = \{1, \ldots, k\}$ and d^k if an element d occurs k times in a multiset. For example, $\{1, 1, 1, 2, 5, 5\} = \{1^3, 2, 5^2\}$.

For a vertex v in G, the *eccentricity* of v, denoted by $ecc(v)$, is the maximum distance from v to any other vertex in G. A vertex u in G is said to be *diametral* if there exists a vertex v in G such that the distance between u and v is equal to $diam(G)$. In this context, the vertex v is said to be *diametral with respect to* u. For any vertex u in G, the *open neighbourhood* of u is the set $N_G(u) = \{v \in V(G) | uv \in E(G)\}$, while the *close neighbourhood* of u is defined by $N_G[u] = N_G(u) \bigcup \{u\}$. The *degree* of a vertex u is given by $|N_G(u)|$. Two vertices u and v in G are called *true twins* if $N_G[u] = N_G[v]$ and *false twins* if $N_G(u) = N_G(v)$. More generally, u and v are referred to as *twins* if they are either true twins or false twins. The existence of twins is important for determining the outer multiset resolving set, as shown below.

Proposition 2.1. *[2] Let G be a nontrivial graph, and $S \subseteq V(G)$ be an outer multiset resolving set of G. If u and v are a pair of twin vertices, then $u \in S$ or $v \in S$.*

We start by determining the outer multiset dimension of stars.

Theorem 2.1. *For $n \geq 3$, $dim_{ms}(S_n = \overline{K_{n-1}} + K_1) = n - 2$.*

Proof. Let $V(K_1) = \{v\}$ and $V(\overline{K_{n-1}}) = \{v_1, v_2, \ldots, v_{n-1}\}$. Thus, $ecc(v) = 1$ and $ecc(v_i) = 2$, for $1 \leq i \leq n - 1$.

From Corollary 1.1, it is sufficient to prove that $dim_{ms}(\overline{K_{n-1}} + K_1) \geq n - 2$. Assume that $S \subseteq V(\overline{K_{n-1}} + K_1)$ is an outer multiset resolving set with $|S| \leq (n - 3)$. We consider two cases: (i) $v \notin S$ and (ii) $v \in S$.

Case (i) ($v \notin S$): There exist distinct $i, j, 1 \leq i, j \leq n - 1$, such that

$$S \subseteq V(\overline{K_{n-1}} + K_1) \setminus \{v, v_i, v_j\}.$$

Thus, v_i and v_j have no adjacent vertices in S and it follows that $m(v_i|S) = \{2^{|S|}\} = m(v_j|S)$.

Case (ii) ($v \in S$): There exist distinct $i, j, k, 1 \leq i, j, k \leq n - 1$, such that

$$S \subseteq V(\overline{K_{n-1}} + K_1) \setminus \{v_i, v_j, v_k\}.$$

Thus, v_i, v_j has only one adjacent vertex in S, that is, v. Then $m(v_i|S) = \{1, 2^{|S|-1}\} = m(v_j|S)$.

In both cases, S is not an outer multiset resolving set. So, $dim_{ms}(\overline{K_{n-1}} + K_1) \geq n - 2$, and this completes the proof. □

Notice that the outer multiset dimension of stars in Theorem 2.1 is equal to the upper bound in Corollary 1.1. In the next two theorems and a corollary, we provide three other families of graphs whose outer multiset dimensions are equal to the upper bound. Thus, in proving the theorems, we are only required to provide that the outer multiset dimensions are at least $n(G) - 2$.

Theorem 2.2. *For $n \geq 4$ and $m \geq 1$,*

$$dim_{ms}(W_{m,n} = C_n + \overline{K_m}) = \begin{cases} m+n-2, & \text{if } m \neq n-2, \\ m+n-1, & \text{if } m = n-2. \end{cases}$$

Proof. For $m = n-2$, the generalized wheel $W_{m,n}$ is n-regular with diameter 2, and so by Theorem 1.2, $dim_{ms}(W_{m,n}) = m+n-1$.

For $m \neq n-2$, let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(\overline{K_m}) = \{u_1, u_2, \ldots, u_m\}$. Thus, $ecc(u_i) = ecc(v_j) = 2$, for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Assume that $S \subseteq V(C_n + \overline{K_m})$ is an outer multiset resolving set with $|S| \leq m+n-3$. Since $\{u_1, u_2, \ldots, u_m\}$ are twin vertices in $C_n + \overline{K_m}$, by Proposition 2.1, every outer multiset resolving set must contain at least $(m-1)$ such vertices. Without loss of generality, let $\{u_1 \ldots, u_{m-1}\} \subseteq S$.

We consider two cases: (i) $u_m \notin S$ and (ii) $u_m \in S$.

Case (i) ($u_m \notin S$): There exist distinct $j, k, 1 \leq j, k \leq n$, such that

$$S \subseteq V(C_n + \overline{K_m}) \setminus \{u_m, v_j, v_k\}.$$

Thus, v_j and v_k have adjacent vertices in S and it follows that $m(v_j|S) = \{1^x, 2^{|S|-x}\} = m(v_k|S)$, where $x = \begin{cases} m, & \text{if } v_j \text{ is adjacent to } v_k, \\ m+1, & \text{if } v_j \text{ is not adjacent to } v_k. \end{cases}$

Case (ii) ($u_m \in S$): There exist distinct $j, k, l, 1 \leq j, k, l \leq n$, such that

$$S \subseteq V(C_n + \overline{K_m}) \setminus \{v_j, v_k, v_l\}.$$

Thus, v_j, v_k have adjacent vertices in S and it follows that $m(v_j|S) = \{1^x, 2^{|S|-x}\} = m(v_k|S)$, where $x = \begin{cases} m+2, & \text{if } v_j \text{ is not adjacent to } v_k \text{ and } v_l \text{ is not adjacent to both } v_j \text{ and } v_k \\ m+1, & \text{otherwise.} \end{cases}$

In both cases, S is not an outer multiset resolving set. Thus, $dim_{ms}(C_n+\overline{K_m}) \geq m+n-2$. □

As a special case of Theorem 2.2, for $m = 1$, we obtained the outer multiset dimension of wheels.

Corollary 2.1. *For $n \geq 5$, $dim_{ms}(W_n = C_{n-1} + K_1) = n-2$.*

Theorem 2.3. *For $n \geq 2$ and $k \geq 2$, $dim_{ms}(W(k,n) = nK_k + K_1) = nk-1$.*

Proof. Let $V(nK_k) = \{h_{(1,1)}, \ldots, h_{(1,k)}, \ldots, h_{(n,1)}, \ldots, h_{(n,k)}\}$ and $V(K_1) = \{v\}$.

Assume that $S \subseteq V(K_1 + nK_k)$ is an outer multiset resolving set with $|S| \leq nk-2$. For each $1 \leq i \leq n$, $h_{(i,a)}$ and $h_{(i,b)}$, $1 \leq a, b \leq k$, are twins. Thus, by Proposition 2.1, any outer multiset resolving set contains at least $k-1$ of them. Without loss of generality, let $\{h_{(1,1)}, \ldots, h_{(1,k-1)}, \ldots, h_{(n,1)}, \ldots, h_{(n,k-1)}\} \subseteq S$.

We consider two cases: (i) $v \notin S$ and (ii) $v \in S$.

Case (i) ($v \notin S$)**:** There exist distinct $i, j, 1 \leq i, j \leq n$, such that

$$S \subseteq V(K_1 + nK_k) \setminus \{v, h_{(i,k)}, h_{(j,k)}\}.$$

It follows that $m(h_{(i,k)}|S) = \{1^{k-1}, 2^{(n-1)(k-1)+n-2}\} = m(h_{(j,k)}|S)$.

Case (ii) $(v \in S)$**:** There exist distinct $i, j, k, 1 \leq i, j, k \leq n$, such that $S \subseteq V(K_1 + nK_k) \setminus \{h_{(i,k)}, h_{(j,k)}, h_{(l,k)}\}$. It follows that $m(h_{(i,k)}|S) = \{1^k, 2^{(n-1)(k-1)+n-3}\} = m(h_{(j,k)}|S)$. Thus, S is not an outer multiset resolving set and $dim_{ms}(nK_k + K_1) \geq nk - 1$. □

In the next two theorems, we provide families of graphs whose outer multiset dimensions are close to the upper bound of Corollary 1.1.

Theorem 2.4. *For* $n \geq 4$*,*

$$dim_{ms}(F_n = P_{n-1} + K_1) = \begin{cases} 2, & \text{if } n = 4, 5, \\ 3, & \text{if } n = 6, \\ n - 4, & \text{if } n \geq 7. \end{cases}$$

Proof. Let $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}, E(P_{n-1}) = \{v_1v_2, \ldots, v_{n-2}v_{n-1}\}$, and $V(K_1) = \{v\}$, Thus, $ecc(v) = 1$ and $ecc(v_i) = 2$, for $1 \leq i \leq n - 1$.

Let S be an outer multiset resolving set. We distinguish three cases:

Case 1 ($n = 4, 5$**):** From Theorem 1.1, $|S| \geq 2$, and so it is sufficient to prove that $|S| \leq 2$. Choose $S = \{v_1, v_2\}$. For $n = 4$, $m(v_3|S) = \{1, 2\}, m(v|S) = \{1^2\}$ and for $n = 5$, $m(v_4|S) = \{2^2\}$. Thus, S is an outer multiset resolving set.

Case 2 ($n = 6$**):** First, assume that $|S| < 3$. By Theorem 1.1, $|S| = 2$. Consider two sub-cases (i) $v \in S$ and (ii) $v \notin S$.

Sub-case 2(i) $(v \in S)$**:** $S = \{v, v_a\}$, for $1 \leq a \leq n-1$. There exist two distinct vertices v_i, v_j, $1 \leq i, j \leq (n - 1)$ with distance two from v_a, and so $m(v_j|S) = m(v_i|S) = \{1, 2\}$.

Sub-case 2(ii) $(v \notin S)$**:** $S = \{v_a, v_b\}$, for $1 \leq a, b \leq n - 1$. When v_a is adjacent to v_b and $a \neq 1$ and $b \neq 5$, then $m(v_{a-1}|S) = \{1, 2\} = m(v_{b+1}|S)$. Otherwise, $a = 1$ or $b = 5$. For $a = 1$, $m(v_4|S) = \{2^2\} = m(v_5|S)$ and for $b = 5$, $m(v_1|S) = \{2^2\} = m(v_2|S)$. When v_a is not adjacent to v_b, then $b - a \geq 1$. If $b - a = 1$, then $m(v_{a+1}|S) = \{1^2\} = m(v|S)$. Otherwise, $b - a \geq 2$, and $m(v_{a+1}|S) = \{1, 2\} = m(v_{b-1}|S)$.

Thus, S is not an outer multiset resolving set, and so $|S| \geq 3$.

Now we prove $|S| \leq 3$, considering $S = \{v_1, v_3, v_4\}$. In this case, $m(v_2|S) = \{1^2, 2\}$, $m(v_5|S) = \{1, 2^2\}$, $m(v|S) = \{1^3\}$. Thus, every vertex outside of S has a different multiset representation.

Case 3 ($n \geq 7$**):** Assume that $|S| \leq n-5$. Let v_i, v_j, v_k, v_l be four distinct vertices not in S. For $t \in \{i, j, k, l\}$, let D_t be the number of adjacent vertices in S adjacent to v_t. Since v_t is adjacent to at most three other vertices in F_n, including v, then $D_t \in \{0, 1, 2\}$, when $v \notin S$, or $D_t \in \{1, 2, 3\}$, when $v \in S$. Since there are four vertices, according to the Pigeonhole principle, there are two vertices, say v_i and v_j, where $D_i = D_j$. If $D_i = D_j = 0$, then $m(v_i|S) = \{2^{|S|}\} = m(v_j|S)$. If $1 \leq D_i = D_j \leq 3$, then $m(v_i|S) = \{1^{D_i}, 2^{|S|-D_i}\} = m(v_j|S)$. Thus, S is not an outer multiset resolving set and $|S| \geq (n - 4)$.

Now we prove $|S| \leq n - 4$, by defining $S = \{v_1, v_3, v_4 \ldots, v_{n-3}\}$. Here, v, v_2, v_{n-2}, and v_{n-1} are distinct vertices outside of S, with

$$m(v_2|S) = \{1^2, 2^{|S|-2}\}, m(v_{n-2}|S) = \{1^1, 2^{|S|-1}\},$$
$$m(v_{n-1}|S) = \{2^{|S|}\}, \text{ and } m(v|S) = \{1^{|S|}\}.$$

Since every vertex outside of S has a different multiset representation, S is an outer multiset resolving set. This completes the required result. □

The next theorem somewhat generalizes the result in Theorem 2.4, in the sense that Theorem 2.5 does not cover $m = 1$, which is the case in Theorem 2.4.

Theorem 2.5. *For $n \geq 2$ and $m \geq n$,*

$$dim_{ms}(F_{m,n} = P_n + \overline{K}_m) = \begin{cases} m, & \text{if } n = 2, 3 \\ m + 1, & \text{if } n = 4 \\ m + n - 4, & \text{if } n \geq 5 \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(\overline{K_m}) = \{u_1, u_2, \ldots, u_m\}$. Thus, $ecc(u_i) = ecc(v_j) = 2$, for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Let S be an outer multiset resolving set. Since $u_1, u_2, \ldots, u_m$ are twin vertices in $P_n + \overline{K_m}$, by Proposition 2.1, S must contain at least $m - 1$ of these vertices. Without loss of generality, let $\{u_1, \ldots, u_{m-1}\} \subseteq S$.

We distinguish three cases:

Case 1 ($n = 2, 3$)**:** Assume $|S| = m - 1$. Then, $m(v_j|S) = m(v_k|S)$, for $1 \leq j, k \leq n$ and $j \neq k$. Thus, S is not an outer multiset resolving set, and so $|S| \geq (m - 1) + 1$.

Now we prove $|S| \leq (m - 1) + 1$, by choosing $S = \{u_1, u_2, \ldots, u_{m-1}\} \bigcup \{v_1\}$. For $n = 2$, $m(v_2|S) = \{1^m\}$ and $m(u_m|S) = \{1, 2^{m-1}\}$. For $n = 3$, $m(v_3|S) = \{1^{m-1}, 2\}$. Thus, S is an outer multiset resolving set.

Case 2 ($n = 4$)**:** Assume $|S| \leq m$. Consider two subcases (i) $u_m \in S$ and (ii) $u_m \notin S$.

Sub-case 2(i) ($u_m \in S$)**:** $m(v_j|S) = \{1^m\} = m(v_k|S)$, for $1 \leq j, k \leq 4$ and $j \neq k$.

Sub-case 2(ii) ($u_m \notin S$)**:** There exists $1 \leq j \leq 4$, such that $v_j \in S$. In this case, there exist two distinct vertices v_k, v_l outside of S having the same distance to $u_1, u_2, \ldots, u_{m-1}$ and v_j, and so $m(v_k|S) = m(v_l|S)$. Thus, S is not an outer multiset resolving set, and $|S| \geq (m - 1) + 2$.

Now we prove $|S| \leq (m - 1) + 2$, by choosing $S = \{u_1, u_2, \ldots, u_{m-1}\} \bigcup \{v_1, v_2\}$. Here, $m(v_3|S) = \{1^m, 2\}$, $m(v_4|S) = \{1^{m-1}, 2^2\}$ and $m(u_m|S) = \{1^2, 2^{m-1}\}$. Thus, S is an outer multiset resolving set.

Case 3 ($n \geq 5$)**:** Assume $|S| \leq m + n - 5$ and let v_j, v_k, v_l, v_p be the four distinct vertices not in S. For $t \in \{j, k, l, p\}$, let D_t be the number of adjacent vertices in S adjacent to v_t. Since v_t is adjacent to at most $m + 2$ other vertices in $F_{(m,n)}$, including u_m, then $D_t \in \{m - 1, m, m + 1\}$, when $u_m \notin S$, or $D_t \in \{m, m + 1, m + 2\}$, when $u_m \in S$. Since there are four vertices, according to the Pigeonhole principle, there are two vertices, say v_j and v_k, where $D_j = D_k$. If $D_j = D_k = m - 1$, then $m(v_j|S) = \{1^{m-1}, 2^{|S|-(m-1)}\} = m(v_k|S)$. If $m \leq D_j = D_k \leq m + 2$, then $m(v_i|S) = \{1^{D_i}, 2^{|S|-D_i}\} = m(v_j|S)$. Thus, S is not an outer multiset resolving set and $|S| \geq m + n - 4$.

Now we prove $|S| \leq m + n - 4$, by defining

$$S = \{u_1, u_2, \ldots, u_{m-1}\} \bigcup \{v_1, v_3, v_4, \ldots, v_{n-2}\}.$$

Here, u_m,v_2,v_{n-1},v_n are distinct vertices outside of S. Then

$$m(v_2|S) = \{1^{m+1}, 2^{n-5}\}, m(v_{n-1}|S) = \{1^m, 2^{n-4}\},$$
$$m(v_n|S) = \{1^{m-1}, 2^{n-3}\}, \text{ and } m(u_m|S) = \{1^{n-3}, 2^{m-1}\}.$$

Every vertex outside of S has a distinct multiset representation, and this completes the required result. □

From all the graphs being studied, the outer multiset dimensions of these graphs are either the same as the upper bound of Corollary 1.1 or one or two less than the bound. Thus, it is natural to ask whether there are joined graphs with outer multiset dimensions "far" from the upper bound of Corollary 1.1, and subsequently what the "good" lower bounds are for $dim_{ms}(G + K_1)$ and $dim_{ms}(G + \overline{K_m})$.

Problem 1. *Find an infinite family of connected graphs G where the outer multiset dimension of $G + K_1$ or $G + \overline{K_m}$ is much less than $n(G)$, in particular, $dim_{ms}(G + K_1) < \frac{n(G)}{2}$ or $dim_{ms}(G + \overline{K_m}) < \frac{n(G)}{2}$.*

Problem 2. *Let G be an arbitrary connected graph. Determine a lower bound for $dim_{ms}(G + K_1)$ and $dim_{ms}(G + \overline{K_m})$.*

We conclude by asking the following two general questions.

Problem 3. *Let G be an arbitrary connected graph, determine $dim_{ms}(G + K_1)$.*

Problem 4. *Let G be an arbitrary connected graph, determine $dim_{ms}(G + \overline{K_m})$.*

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