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Local edge antimagic chromatic number of join product of graphs

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Abstract

Let $f : V(G) \to [1, |V(G)|]$ be a bijective mapping of the vertex set of a graph G to the integers 1 through $|V(G)|$. A labeling f is defined as a local edge antimagic labeling if, for any two adjacent edges uv and vx in $E(G)$, their weights satisfy $w_f(uv) \neq w_f(vx)$, where the weight of an edge uv is given by $w_f(uv) = f(u) + f(v)$. The weight w_f induces a proper edge coloring on G. The local edge antimagic chromatic number of G, denoted $\chi'_{lea}(G)$, is the minimum number of colors required among all colorings induced by local edge antimagic labelings of G. In this paper, we investigate the local edge antimagic coloring of join product of graphs, particularly for independent sets, paths, and cycles.

Keywords: edge coloring, local edge antimagic, join product
Mathematics Subject Classification : 05C78, 05C15

1. Introduction

The concept of magic graphs originated from the work of Kotzig and Rosa in 1970, who introduced magic valuations. In this framework, a graph is assigned edge labels such that the sum of edge labels incident to each vertex (often including vertex labels) is the same for all vertices [11]. This idea, which aligns with the concept of a ”magic square” applied to graphs, opened a new avenue in combinatorial mathematics. The study of magic graphs became a foundational topic in graph labeling, inspiring further exploration into variations such as vertex-magic and edge-magic graphs.

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Enomoto et al. expanded the field in 1998 by introducing (super) edge-magic graphs. These graphs assign a bijection of labels to vertices and edges such that a constant sum arises when combining edge labels with their incident vertex labels [5]. This extension refined the study of magic graphs, paving the way for deeper combinatorial investigations.

Antimagic graphs, proposed later, flip the focus from uniformity to uniqueness. A graph is antimagic if its edges can be labeled with distinct integers such that the sum of edge labels at each vertex is distinct. The formalization of this concept addressed combinatorial configurations that avoid symmetry, offering a counterpoint to magic graphs.

Significant contributions to this area were made by Bača et al., who in the early 2000s provided new constructions for magic and antimagic graphs [2]. In 2007, they expanded their focus to edge-antimagic graphs, emphasizing configurations where distinct sums are achieved at the vertices [3]. Antimagic graphs have since been studied in various graph families and operations.

The notion of local edge antimagic (LEA) graphs represents a localized extension of antimagic labeling. Instead of focusing on the entire graph, this concept looks at pairs of edges. Specifically, a graph is LEA if there exists a vertex labeling such that the sum of vertex labels incident to each edges forms a proper edge coloring (i.e., distinct sums for adjacent edges).

The formal study of LEA labeling began with Agustin et al. in 2017. Their work defined the concept and provided examples for specific graph families, such as paths, cycles, ladders, stars, complete graph [1]. Moreover, Rajkumar and Nalliah [13] investigated even further by considering several class of graphs, such as friendship graphs, wheels, fan graphs, and helm graphs. A characterization of small number of graphs with small LEA chromatic numbers and bounds of LEA chromatic number for any graphs were given in [7]. Most recently, Chandra and Silaban [4] determined the LEA chromatic numbers for certain comb products involving path graphs. A study of graphs which attains the highest LEA chromatic number is also conducted in [10]. Variations of local antimagic labeling can be seen in [8, 9, 12, 14] and information about graph labeling in general is provided in [6].

In this paper, we determine the LEA chromatic number of join product of graphs. First, we present the LEA coloring of complete bipartite graph, a special case of join product, which gives a basic understanding of LEA coloring in join products. Next, we provide a bound of LEA chromatic number for the join product of any graphs. In addition, we investigated the LEA chromatic number of the join product among paths, cycles, and independent sets.

2. Main Results

Let $[a, b]$ be consecutive integers from a up to b inclusively. Let $\Delta(G)$ be the maximum degree of a vertex among all vertices of G and let $\chi'(G)$ be the chromatic index of the graph G.

Let $f : V(G) \to [1, |V(G)|]$ be a bijective map. We call f to be *local edge antimagic (LEA)* labeling if for every adjacent edges uv and vx in $E(G)$, it holds that $w_f(uv) \neq w_f(vx)$ where $w_f(uv) = f(u) + f(v)$. The LEA chromatic number of G, $\chi'_{lea}(G)$, is the least number of colors taken over all edge-colorings induced by LEA labeling of G. It is known that all graphs admit a LEA labeling [7]. In this study, it is safe to assume that graphs without edges have zero colors, or equivalently $\chi'_{lea}(\overline{K_m}) = 0$ for any positive integer m. It is evident that the following inequality holds:

$$\Delta(G) \leq \chi'(G) \leq \chi'_{lea}(G) \leq |E(G)|.$$

Let G and H be undirected, finite and simple graphs. A *join product* of G and H, denoted by $G + H$, is a graph defined by the vertex set

$$V(G + H) = V(G) \cup V(H)$$

and the edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}.$$

The join product of any two graph is always connected. The LEA chromatic number of several special cases of join products have been determined. We write these results with an additional form in join product.

Theorem 2.1. *[13] For the friendship graph $F_n \cong K_1 + nK_2$, we have*

$$\chi'_{lea}(F_n) = \begin{cases} 3, & \text{if } n = 1, \\ 2n, & \text{if } n \geq 2. \end{cases}$$

Theorem 2.2. *[13] For the wheel graph $W_n \cong K_1 + C_n$, we have*

$$\chi'_{lea}(W_n) = \begin{cases} 5, & \text{if } n = 3, 4, \\ n, & \text{if } n \geq 5. \end{cases}$$

Theorem 2.3. *[13] For the fan graph $T_n \cong K_1 + P_n$, we have*

$$\chi'_{lea}(T_n) = \begin{cases} n + 1, & \text{if } n = 2, 3, \\ n, & \text{if } n \geq 4. \end{cases}$$

First, we will deal with complete bipartite graphs $K_{m,n}$. It is known that this is another special case of join product of two independent sets where $K_{m,n} \cong \overline{K_m} + \overline{K_n}$.

Theorem 2.4. *Let m and n be positive integers. It follows that $\chi'_{lea}(K_{m,n}) = m + n - 1$.*

Proof. Let $K_{m,n}$ be defined by the vertex set $V(K_{m,n}) = \{u_i, v_j \mid i \in [1, m], j \in [1, n]\}$ and the edge set $E(K_{m,n}) = \{u_i v_j \mid i \in [1, m], j \in [1, n]\}$.

To show $\chi'_{lea}(K_{m,n}) \geq m + n - 1$, let f be arbitary LEA labeling of $K_{m,n}$. Without loss of generality, we can consider only the case of f where $f(u_i) < f(u_{i+1})$ and $f(v_j) < f(v_{j+1})$ for $i \in [1, m - 1]$ and $j \in [1, n - 1]$. It follows that

$$\begin{aligned} f(u_i) &< f(u_{i+1}) \\ f(u_i) + f(v_j) &< f(u_{i+1}) + f(v_j) \\ w_f(u_i v_j) &< w_f(u_{i+1} v_j) \end{aligned}$$

and similarly we can deduce $w_f(u_iv_j) < w_f(u_iv_{j+1})$. This implies

$$w_f(u_1v_1) < w_f(u_1v_2) < \cdots < w_f(u_1v_n) < w_f(u_2v_n) < \cdots < w_f(u_mv_n)$$

or equivalently, we have $m+n-1$ distinct colors in $K_{m,n}$. Therefore, $\chi'_{lea}(K_{m,n}) \geq m+n-1$.

Next, we will show $\chi'_{lea}(K_{m,n}) \leq m+n-1$. Define a map $f : V(G) \to [1, m+n]$ as follows:

$$f(u_i) = i, \qquad \text{for } i \in [1,m],$$
$$f(v_j) = m+j, \qquad \text{for } j \in [1,n].$$

Then, we have the following induced weights w_f of f:

$$w_f(u_iv_j) = i+j+m, \qquad \text{for } i \in [1,m], j \in [1,n].$$

It is not hard to deduce that w_f will induce $m+n-1$ colors. Hence, $\chi'_{lea}(K_{m,n}) \leq m+n-1$. □

A consequence of the preceding theorem is that $\chi'_{lea}(G) - \chi'(G)$ can be arbitrarily large.

Corollary 2.1. *Let a and b be positive integers where $a \leq b \leq 2a-1$. There exists a graph G with $\chi'(G) = a$ and $\chi'_{lea}(G) = b$.*

Proof. Consider $G = K_{a,b-a+1}$ and apply Theorem 2.4. □

An example of Theorem 2.4 is given in Fig. 1.

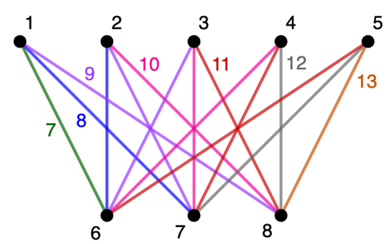

Figure 1: The LEA coloring of $K_{3,5}$ with $\chi'_{lea}(K_{3,5}) = 7$. All edges with the same weight are given the same colors.

Furthermore, we can utilize Theorem 2.4 to provide bounds for any join product of two graphs.

Theorem 2.5. *Let G and H be graphs. We have*

(1) $\chi'_{lea}(G+H) \geq |V(G)| + |V(H)| - 1$,
(2) $\chi'_{lea}(G+H) \leq \chi'_{lea}(G) + \chi'_{lea}(H) + |V(G)| + |V(H)| - 1$.

Proof. Let $m = |V(G)|$ and $n = |V(H)|$. To show (1), assume there exists a LEA labeling f of $G+H$ with k colors where $k < m+n-1$. If we delete all edges in G and H, then f is a LEA labeling of $K_{m,n}$ with k colors. However, this is a contradiction to Theorem 2.4 since $\chi'_{lea}(K_{m,n}) = m+n-1$.

To show (2), let g be a LEA labeling of G and h be a LEA labeling of H. Define $f : V(G+H) \to [1, m+n]$ such that

$$f(v) = \begin{cases} g(v), & \text{if } v \in V(G), \\ h(v) + m, & \text{if } v \in V(H). \end{cases}$$

Then, we have the induced weight map

$$w_f(uv) = \begin{cases} w_g(uv), & \text{if } u, v \in V(G), \\ f(u) + h(v) + m, & \text{if } u \in V(G) \text{ and } v \in V(H), \\ w_h(uv) + 2m, & \text{if } u, v \in V(H). \end{cases}$$

Here, all edges in G will use at most $\chi'_{lea}(G)$ colors and all edges in H will use at most $\chi'_{lea}(H)$ colors. Meanwhile, if we delete all edges in G and H, then f is a LEA labeling of $K_{m,n}$. This implies that there are $m + n - 1$ colors that are connecting G and H. If the set of colors used in G, H and in between are disjoint, then at the worst case we have $\chi'_{lea}(G) + \chi'_{lea}(H) + m + n - 1$ colors. This shows (2). □

Before we continue, we provide a labeling of a graph which we will utilize in the last two theorem. Let $n_1, n_2 \geq 2, m_1, m_2 \geq 1$ be non-negative integers. Let G be a graph with the vertex set

$$V(G) = \{u_{i_1}, u'_{j_1}, v_{i_2}, v'_{j_2} \mid i_1 \in [1, m_1], j_1 \in [1, n_1], i_2 \in [1, m_2], j_2 \in [1, n_2]\}$$

and any edge set $E(G)$. Define a labeling f as follows.

$$\begin{aligned} f(u_{i_1}) &= m_1 - i_1 + 1, \\ f(u'_{j_1}) &= \begin{cases} m_1 + \frac{j_1+1}{2}, & \text{if } j_1 \text{ is odd}, \\ m_1 + n_1 - \frac{j_1}{2} + 1, & \text{if } j_1 \text{ is even}, \end{cases} \\ f(v_{i_2}) &= m_1 + n_1 + n_2 + i_2, \\ f(v'_{j_2}) &= \begin{cases} m_1 + n_1 + n_2 - \frac{j_2+1}{2} + 1, & \text{if } j_2 \text{ is odd}, \\ m_1 + n_1 + \frac{j_2}{2}, & \text{if } j_2 \text{ is even}. \end{cases} \end{aligned} \tag{1}$$

Now, the lower bound provided in Theorem 2.5 is sharp. Other than $\overline{K_m} + \overline{K_n}$, we also have several families of join products.

Theorem 2.6. *Let m_1, m_2, n_1 and n_2 be non-negative integers. We have $\chi'_{lea}(G) = m_1 + m_2 + n_1 + n_2 - 1$ if G is one of the following form:*

(1) $G \cong (\overline{K_{m_1}} \cup P_{n_1}) + (\overline{K_{m_2}} \cup P_{n_2})$, where $n_1, n_2 \geq 2, m_1, m_2 \geq 1$,
(2) $G \cong (\overline{K_{m_1}} \cup P_{n_1}) + (\overline{K_{m_2}} \cup C_{n_2})$, where $n_1 \geq 2, n_2 \geq 3, m_1 \geq 1, m_2 \geq \frac{n_2+1}{2}$,
(3) $G \cong (\overline{K_{m_1}} \cup C_{n_1}) + (\overline{K_{m_2}} \cup C_{n_2})$, where $n_1, n_2 \geq 3, m_1 \geq \frac{n_1+1}{2}, m_2 \geq \frac{n_2+1}{2}$,
(4) $G \cong \overline{K_{m_1}} + (\overline{K_{m_2}} \cup P_{n_2})$, where $n_1 = 0, n_2 \geq 2, m_1, m_2 \geq 1$,

(5) $G \cong \overline{K_{m_1}} + (\overline{K_{m_2}} \cup C_{n_2})$*, where* $n_1 = 0, n_2 \geq 3, m_1 \geq 1, m_2 \geq \frac{n_2+1}{2}$.

Proof. First, we will show (1). Let $n_1, n_2 \geq 2, m_1, m_2 \geq 1$ be positive integers. Let $G = (\overline{K_{m_1}} \cup P_{n_1}) + (\overline{K_{m_2}} \cup P_{n_2})$, be a graph defined by the vertex set

$$V(G) = \{u_{i_1}, u'_{j_1}, v_{i_2}, v'_{j_2} \mid i_1 \in [1, m_1], j_1 \in [1, n_1], i_2 \in [1, m_2], j_2 \in [1, n_2]\}$$

and the edge set

$$\begin{aligned} E(G) = \{&u_{i_1}v_{i_2}, u_{i_1}v'_{j_2}, \mid i_1 \in [1, m_1], i_2 \in [1, m_2], j_2 \in [1, n_2]\} \\ &\cup \{u'_{j_1}v_{i_2}, u'_{j_1}v'_{j_2} \mid j_1 \in [1, n_1], i_2 \in [1, m_2], j_2 \in [1, n_2]\} \\ &\cup \{u'_{j_1}u'_{j_1+1}, v'_{j_2}v'_{j_2+1} \mid j_1 \in [1, n_1 - 1], j_2 \in [1, n_2 - 1]\}. \end{aligned}$$

Let G be applied a labeling f defined in eq. (1). Let w_f be the induced weight function. We know that there would be $m_1 + n_1 + m_2 + n_2 - 1$ colors in the set of edges connecting G and H. Now, if j_1 is odd, then

$$w_f(u'_{j_1}u'_{j_1+1}) = 2m_1 + n_1 + 1 = w_f(u_1v'_2).$$

Otherwise, if j_1 is even, it holds that

$$w_f(u'_{j_1}u'_{j_1+1}) = 2m_1 + n_1 + 2 = \begin{cases} w_f(u_1v'_4), & \text{if } n_2 \geq 4, \\ w_f(u_1v'_3), & \text{if } n_2 = 3, \\ w_f(u_1v'_1), & \text{if } n_2 = 2. \end{cases}$$

Likewise, if j_2 is odd, we have

$$w_f(v'_{j_2}v'_{j_2+1}) = 2m_1 + 2n_1 + n_2 + 1 = w_f(u'_2v_1),$$

and if j_2 is even, it follows that

$$w_f(v'_{j_2}v'_{j_2+1}) = 2m_1 + 2n_1 + n_2 = \begin{cases} w_f(u'_4v_1), & \text{if } n_1 \geq 4, \\ w_f(u'_3v_1), & \text{if } n_1 = 3, \\ w_f(u'_1v_1), & \text{if } n_1 = 2. \end{cases}$$

Therefore, w_f induces only $m_1 + m_2 + n_1 + n_2 - 1$ colors. Consequently, $\chi'_{lea}(G) = m_1 + m_2 + n_1 + n_2 - 1$.

From here, the proof will be similar. Now, we will show (2). For positive integers $n_1 \geq 2, n_2 \geq 3, m_1 \geq 1, m_2 \geq \frac{n_2+1}{2}$, let $G = (\overline{K_{m_1}} \cup P_{n_1}) + (\overline{K_{m_2}} \cup C_{n_2})$ be defined simply from $(\overline{K_{m_1}} \cup P_{n_1}) + (\overline{K_{m_2}} \cup P_{n_2})$ with an edge addition of $v'_1v'_{n_2}$. Since they have the same vertex set, we can also apply f in eq. (1) to $V(G)$. If n_2 is odd, it is evident that

$$w_f(v'_1v'_{n_2}) = 2m_1 + 2n_1 + \frac{3n_2 + 1}{2} = w_f(u'_2v_{\frac{n_2+1}{2}}).$$

Otherwise, if n_2 is even, it holds that

$$w_f(v'_1 v'_{n_2}) = 2m_1 + 2n_1 + \frac{3n_2}{2} = w_f(u'_2 v_{\frac{n_2}{2}}).$$

Hence, there are only $m_1 + m_2 + n_1 + n_2 - 1$ colors induced by w_f. This implies $\chi'_{lea}(G) = m_1 + m_2 + n_1 + n_2 - 1$.

To show (3), let $n_1, n_2 \geq 3, m_1 \geq \frac{n_1+1}{2}, m_2 \geq \frac{n_2+1}{2}$ be positive integers. Let $G = (\overline{K_{m_1}} \cup C_{n_1}) + (\overline{K_{m_2}} \cup C_{n_2})$ be defined from $(\overline{K_{m_1}} \cup P_{n_1}) + (\overline{K_{m_2}} \cup C_{n_2})$ with an edge addition of $u'_1 u'_{n_1}$. Again, apply f in eq. (1) to $V(G)$. If n_1 is odd, then we have

$$w_f(u'_1 u'_{n_1}) = 2m_1 + \frac{n_1+3}{2} = w_f(u_{\frac{n_1+1}{2}} v'_2).$$

and also if n_1 is even, it follows that

$$w_f(u'_1 u'_{n_1}) = 2m_1 + \frac{n_1}{2} + 2 = w_f(u_{\frac{n_1}{2}} v'_2).$$

Therefore, $\chi'_{lea}(G) = m_1 + m_2 + n_1 + n_2 - 1$.

The statement (4) is just an implication of (1) by omitting the labeling of the vertices of u_{i_1} and removing some edges appropriately. Similarly, the statement (5) is an implication of (3) by omitting the same labeling of u_{i_1} and removing some edges appropriately. □

Fig. 2 shows an example of LEA coloring of $(\overline{K_1} \cup P_3) + (\overline{K_3} \cup C_4)$.

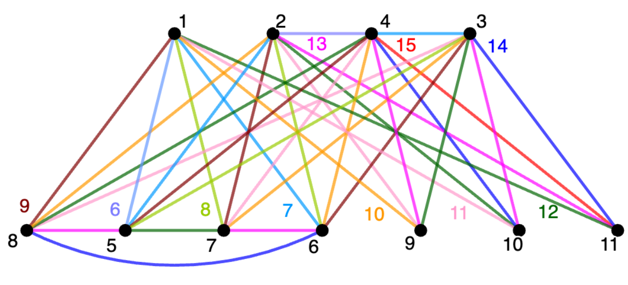

Figure 2: The LEA coloring of $G = (\overline{K_1} \cup P_3) + (\overline{K_3} \cup C_4)$ with $\chi'_{lea}(G) = 10$.

We also have the following bounds for the join product between paths and cycles.

Theorem 2.7. *Let m and n be positive integers. It follows that*

(1) $m + n - 1 \leq \chi'_{lea}(P_m + P_n) \leq m + n + 1$, *where* $m, n \geq 3$,
(2) $m + n - 1 \leq \chi'_{lea}(P_m + C_n) \leq m + n + 2$, *where* $m \geq 2, n \geq 3$,
(3) $m + n - 1 \leq \chi'_{lea}(C_m + C_n) \leq m + n + 3$, *where* $m, n \geq 2$.

Proof. The lower bound in each statement is just an implication of Theorem 2.5. Now, we just need to prove the upper bound. To show the upper bound in (1), let $G = P_m + P_n$ be defined below

$$V(G) = \{u'_i, v'_j \mid i \in [1, m], j \in [1, n]\}.$$

Next, apply f in eq. (1) to G by omitting the vertices u_k and v_k for some k (since $m_1 = 0 = m_2$) and fix $n_1 = m$, $n_2 = n$. The colors of edges connecting P_m and P_n are exactly $m + n - 1$ colors. Now, if $i \in [1, m]$ is odd, then

$$w_f(u'_i u'_{i+1}) = m + 1$$

while if $i \in [1, m]$ is even, it is evident that

$$w_f(u'_i u'_{i+1}) = m + 2 = w_f(u'_1 v'_2).$$

Meanwhile, if $j \in [1, n]$ is odd, then

$$w_f(v'_j v'_{j+1}) = 2m + n + 1$$

and if $j \in [1, n]$ is even, we have

$$w_f(v'_j v'_{j+1}) = 2m + n = w_f(u'_2 v'_1).$$

Therefore, in addition to $m + n - 1$ colors in edges connecting P_m and P_n, there are also the color of $w_f(u'_i u'_{i+1})$ when i is odd and $w_f(v'_j v'_{j+1})$ when j is odd. It follows that $\chi'_{lea}(G) \le m + n + 1$.

The statement (2) is a consequence of (1) by assuming the new edge has a unique color. Similarly, (3) is implied by (2) by assuming again the uniqueness of the new edge. □

To conclude this paper, we would like to propose new open problems that are not covered here.

Problem 1. *Does there exists two graphs G and H such that*

$$\chi'_{lea}(G + H) = \chi'_{lea}(G) + \chi'_{lea}(H) + |V(G)| + |V(H)| - 1?$$

Problem 2. *Characterize graphs G and H which satisfies*

$$\chi'_{lea}(G + H) = |V(G)| + |V(H)| - 1.$$

Problem 3. *Determine the LEA chromatic number $\chi'_{lea}(G + H)$ for any other two graphs G and H.*

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