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On coloring of fractional powers of star, wheel, friendship, and fan graphs

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Abstract

Let G be a simple, connected, and undirected graph. For $m, n \in \mathbb{N}$, the fractional power $G^{\frac{m}{n}} = \left(G^{\frac{1}{n}}\right)^m$ of G is constructed by taking the n-subdivision of G (replacing each edge with a path of length n), and then raising the resulting graph to the m-th power (connecting any two distinct vertices with distance at most m). Let $\omega(G)$ be the clique number of G and $\chi(G)$ be the chromatic number of G. Iradmusa formulated a closed form for the clique number of $G^{\frac{m}{n}}$ ($\omega(G^{\frac{m}{n}})$) and conjectured that $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$ for every $m, n \in \mathbb{N}$ where $\frac{m}{n} < 1$ and $\Delta(G) \geq 3$. The conjecture is true for certain special cases, such as paths, cycles, and complete graphs. However, Hartke et. al. found a counterexample to the conjecture, that is the graph $G = C_3 \Box K_2$ when $m = 3$ and $n = 5$. In this paper, we aim to prove that the conjecture is true for some classes of graphs that have not yet been addressed. We prove that $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$ for star, wheel, friendship, and fan graphs G.

Keywords: graph fractional power, chromatic number, clique number, star graph, wheel graph, friendship graph, fan graph
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1. Introduction

All graphs $G = (V, E) = (V(G), E(G))$ considered in this paper are simple, connected, and undirected. For any graph G and $n \in \mathbb{N}$, as in [8], we define the n-*subdivision* of G, denoted by $G^{\frac{1}{n}}$, to be the graph obtained from G by replacing every edge of G with a path of length n called a *hyperedge*. For every edge $uv \in E(G)$, the hyperedge $ux^1x^2 \dots x^{n-1}v$, or simply uv of $G^{\frac{1}{n}}$, consists of the "old" vertices u, v, called *terminal vertices*, and the "new" vertices $x^1, x^2, \dots, x^{n-1}$ called *internal vertices*. Also, as in [1], for $m \in \mathbb{N}$, the m-*power* of G, denoted G^m, is the graph defined on the same set of vertices as G with an edge between any two vertices u, v iff their distance in G is at most m. Formally, $V(G^m) = V(G)$ and $E(G^m) = \{xy : x, y \in V(G^m), 1 \leq d_G(x, y) \leq m\}$. Finally, following [7], for $\frac{m}{n} < 1$ (that is, $m < n$), we define $G^{\frac{m}{n}} = \left(G^{\frac{1}{n}}\right)^m$, which is called a *fractional power* of G.

For any graph G, let $\chi(G)$ be its chromatic number, $\omega(G)$ be its clique number, and $\Delta(G)$ be its maximum degree. The *chromatic number* of a graph G is the minimum number of colors needed for a vertex coloring of G [9]. A *clique* in a graph G is a complete subgraph of G and the *clique number* of G is the order of its largest clique [4]. On the vertex coloring of fractional powers of graphs, Iradmusa conjectured in [7] that $\chi\left(G^{\frac{m}{n}}\right) = \omega\left(G^{\frac{m}{n}}\right)$ must hold for every $m, n \in \mathbb{N}$ whenever $\frac{m}{n} < 1$ and $\Delta(G) \geq 3$. However, in [5], Hartke et. al. constructed a counterexample to this particular conjecture: the graph $G = C_3 \square K_2$ when $m = 3$ and $n = 5$. The conjecture is then reformulated as follows, simply excluding the counterexample.

Conjecture 1. *If G is a connected graph with $\Delta(G) \geq 3$ and $1 < m < n$, then $\chi\left(G^{\frac{m}{n}}\right) = \omega\left(G^{\frac{m}{n}}\right)$, except when $G = C_3 \square K_2$.*

Recent research includes the establishment of asymptotic bounds for $\chi\left(G^{\frac{m}{n}}\right)$ in cases $\frac{m}{n} = \frac{2}{3}$ and $m = n$ [2], potentially helpful in investigating the equality to $\omega\left(G^{\frac{m}{n}}\right)$. Furthermore, the conjecture above has been proven for several classes of graphs, such as paths and cycles [7] as well as complete graphs [5]. Continuing the quest, we mainly aim to prove the conjecture for star graphs, then wheel graphs. We define the *star graph* S_k to be the graph with $k + 1$ vertices such that the only edges are between one vertex, called the *center*, and every other vertex (each of which is called a *leaf*, and are collectively called the *leaves*) [3].

We similarly define the k-wheel or *wheel graph* W_k to have $k+1$ vertices, being the graph join of a cycle called the *rim* and a vertex called the *hub* [10], i.e. $W_k = C_k + K_1$.

Due to their similarities in structure with wheel graphs, we then aim to prove the conjecture for friendship graphs and fan graphs. As in [6], we define the friendship graph F_k to be constructed by joining k copies of C_3 with a common vertex. Also following [6], we define the fan graph $F_{1,k}$ by $F_{1,k} = K_1 + P_k$.

For star and wheel graphs G, let u be the vertex of G with the highest degree and let x_i be the rest of the vertices of G. In $G^{\frac{1}{n}}$, $x_i^1, x_i^2, x_i^3, ..., x_i^{n-2}, x_i^{n-1}$ are the vertices in the hyperedge $E(ux_i)$ (See Figure 1) and $x_{i,j}^1, x_{i,j}^2, x_{i,j}^3, ..., x_{i,j}^{n-2}, x_{i,j}^{n-1}$ are the vertices in the hyperedge $E(x_ix_j)$ (See Figure 2). Note that using this notation, $x_{i,j}^k$ refers to the same vertex as $x_{j,i}^{n-k}$.

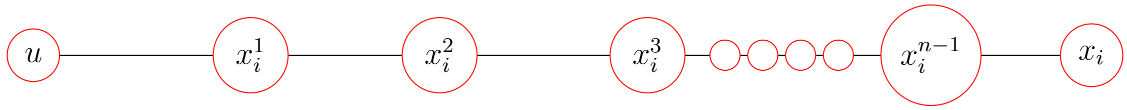

Figure 1: The hyperedge between u and x_i

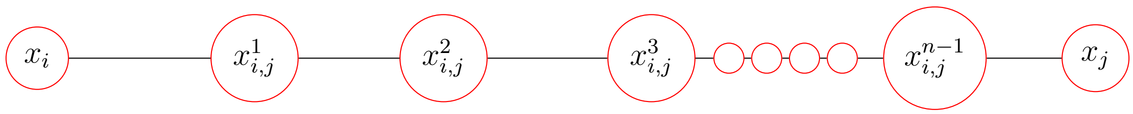

Figure 2: The hyperedge between x_i and x_j

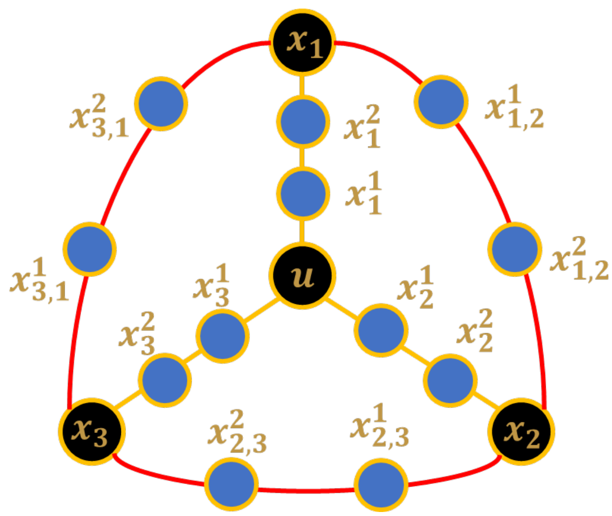

Figure 3: The fractional power of wheel graph, $W_3^{\frac{1}{3}}$

In the following part, we define some terms (from [7]) that we will use in the proofs of our results. For every $x_i \in V(G) \setminus \{u\}$, the bubble in the hyperedge ux_i in $G^{\frac{m}{n}}$ is an ordered $\lfloor \frac{m}{2} \rfloor$-tuple of vertices of $G^{\frac{m}{n}}$ defined by:

$$B_{ux_i} = \left(x_i^1, x_i^2, \ldots, x_i^{\lfloor \frac{m}{2} \rfloor} \right).$$

For every $x_i, x_j \in V(G) \setminus \{u\}$ where $x_i \neq x_j$, the bubble in the hyperedge x_ix_j in $G^{\frac{m}{n}}$ is an ordered $\lfloor \frac{m}{2} \rfloor$-tuple of vertices of $G^{\frac{m}{n}}$ defined by:

$$B_{x_ix_j} = \left(x_{i,j}^1, x_{i,j}^2, \ldots, x_{i,j}^{\lfloor \frac{m}{2} \rfloor} \right).$$

If m is odd, then the crust at vertex u is the set of vertices of $G^{\frac{m}{n}}$ defined as:

$$C_u = \left\{ x_i^{\frac{m+1}{2}} \in V\left(G^{\frac{m}{n}}\right) : ux_i \in E(G) \right\}.$$

Lastly, the middle part of ux_i (M_{ux_i}) and the middle part of x_ix_j ($M_{x_ix_j}$) are the tuple of vertices of $G^{\frac{m}{n}}$ between the two bubbles (or the two crusts if m is odd) on the hyperedges ux_i and x_ix_j, respectively, defined by:

$$M_{ux_i} = \left(x_i^{\lceil \frac{m}{2}+1 \rceil}, \ldots, x_i^{n-\left(\lceil \frac{m}{2}+1 \rceil\right)} \right),$$

$$M_{x_ix_j} = \left(x_{i,j}^{\lceil \frac{m}{2}+1 \rceil}, \ldots, x_{i,j}^{n-\left(\lceil \frac{m}{2}+1 \rceil\right)} \right).$$

2. Known Results

There are several proven theorems that play an important role in our proof. The first one provides a lower bound for the chromatic number of a graph.

Theorem 2.1. *[4] For every graph G of order n, $\chi(G) \geq \omega(G)$.*

Based on Theorem 2.1, the value of $\chi(G)$ is bounded below by the value of $\omega(G)$. Therefore, to determine a lower bound for $\chi\left(G^{\frac{m}{n}}\right)$, we need the value of $\omega\left(G^{\frac{m}{n}}\right)$ that has been found in the following theorem.

Theorem 2.2. *[7] Let G be a graph, $n, m \in \mathbb{N}$ and $m < n$. Then*

$$\omega\left(G^{\frac{m}{n}}\right) = \begin{cases} m+1 & \Delta(G) = 1, \\ \frac{m}{2}\Delta(G) + 1 & \Delta(G) \geq 2, m \equiv 0 \,(mod\, 2), \\ \frac{m-1}{2}\Delta(G) + 2 & \Delta(G) \geq 2, m \equiv 1 \,(mod\, 2). \end{cases}$$

In some references, it has been proven that some classes of graphs fulfill the conjecture. The class of graphs needed in our proof is the complete graphs. The following theorem shows that complete graphs satisfy the conjecture.

Theorem 2.3. *[5] If G is a complete graph, then $\chi\left(G^{\frac{m}{n}}\right) = \omega\left(G^{\frac{m}{n}}\right)$.*

3. Main Results

For each class of graph G that will be considered in this paper, we will show that $\chi\left(G^{\frac{m}{n}}\right) = \omega\left(G^{\frac{m}{n}}\right)$ by finding a vertex coloring for $G^{\frac{m}{n}}$ using $\omega\left(G^{\frac{m}{n}}\right)$ colors, which results in $\chi\left(G^{\frac{m}{n}}\right) \leq \omega\left(G^{\frac{m}{n}}\right)$, thus proving $\chi\left(G^{\frac{m}{n}}\right) = \omega\left(G^{\frac{m}{n}}\right)$ since $\chi\left(G^{\frac{m}{n}}\right) \geq \omega\left(G^{\frac{m}{n}}\right)$ based on Theorem 2.1 above. The rest of the proof is to color one of the maximum cliques in $G^{\frac{m}{n}}$, and then carefully color the other vertices using only colors from the clique.

For notation purposes, we define $k \bmod k = k$ and to simplify the proof, we calculate the index in the modular of k.

3.1. Coloring on star graphs

Theorem 3.1. *Let S_k be a star graph on $k+1$ vertices with $k \geq 3$. Then $\chi\left(S_k^{\frac{m}{n}}\right) = \omega\left(S_k^{\frac{m}{n}}\right)$.*

Proof. We prove the above theorem by coloring star graphs with minimal number of colors. We will find the chromatic number of $S_k^{\frac{m}{n}}$ by finding a vertex coloring using $\omega\left(S_k^{\frac{m}{n}}\right)$ colors for each natural number $k \geq 3$.

From Theorem 2.2,

$$\omega\left(S_k^{\frac{m}{n}}\right) = \begin{cases} \frac{mk}{2} + 1, & m \text{ is even} \\ \frac{(m-1)k}{2} + 2, & m \text{ is odd.} \end{cases}$$

Note that for each $i, j = 1, 2, 3, ..., k$, every vertex in the bubble B_{ux_i} is adjacent to the vertex u and all vertices in the bubble B_{ux_j}. Moreover, when m is odd, every vertex in the crust C_u is adjacent to every vertex in the bubble B_{ux_j} for every $j = 1, 2, 3, ..., k$. Thus, a complete subgraph of order $\omega\left(S_k^{\frac{m}{n}}\right)$ is formed and is a maximum clique.

Each vertex in the maximum clique has to be colored with different colors since they are adjacent to one another. So, the number of colors used so far is $\omega\left(S_k^{\frac{m}{n}}\right)$. Next, we will color other vertices outside of the maximum clique without adding new colors.

Consider the hyperedge ux_i. Since there exist a vertex in the bubble B_{ux_i} that is not adjacent to the vertex outside of the bubble, we can assign the color of such vertices to the vertex outside of the bubble. Notice that the vertex x_i^{m+1} is not adjacent to u and can be colored with the same color as u. Likewise, every vertex x_i^{m+p+1} can be colored the same color as the vertex x_i^p. Continuing this way, we color every vertex in the hyperedge. Applying the same method for other hyperedges, we have a vertex coloring of $S_k^{\frac{m}{n}}$ using $\omega\left(S_k^{\frac{m}{n}}\right)$ colors. The following algorithm describes the method for coloring.

Algorithm 1 Coloring the Graph $S_k^{\frac{m}{n}}$

for $i = 1, 2, ..., k$ **do**
    for $j = 1, 2, ..., n$ **do**
        if $j \leq \left\lfloor \frac{m}{2} \right\rfloor$ **then**
            $x_i^{\left\lceil \frac{m}{2} \right\rceil + j} \leftarrow x_{i+1}^{\left\lfloor \frac{m}{2} \right\rfloor + 1 - j}$
        else if $j = \left\lfloor \frac{m}{2} \right\rfloor + 1$ **then**
            $x_i^{\left\lceil \frac{m}{2} \right\rceil + j} \leftarrow u$
        else if $j > \left\lfloor \frac{m}{2} \right\rfloor + 1$ **then**
            $x_i^{\left\lceil \frac{m}{2} \right\rceil + j} \leftarrow x_i^{j - \left\lceil \frac{m}{2} \right\rceil}$
        end if
    end for
end for

We conclude that the conjecture holds for star graph S_k with $k+1$ vertices for all $k \geq 3$, that is, $\chi\left(S_k^{\frac{m}{n}}\right) = \omega\left(S_k^{\frac{m}{n}}\right)$. □

3.2. Coloring on wheel graphs

Theorem 3.2. *Let W_k be a wheel graph on $k+1$ vertices with $k \geq 3$. Then* $\chi\left(W_k^{\frac{m}{n}}\right) = \omega\left(W_k^{\frac{m}{n}}\right)$.

Proof. We prove the theorem by finding a coloring for wheel graphs with the minimal number of colors. Note that the wheel graph W_3 is isomorphic to K_4 which fulfills the conjecture by Theorem 2.3. We will show that the conjecture holds for W_k with $k \geq 4$.

Based on Theorem 2.2,

$$\omega\left(W_k^{\frac{m}{n}}\right) = \begin{cases} \frac{mk}{2} + 1, & m \text{ is even}, \\ \frac{(m-1)k}{2} + 2, & m \text{ is odd}. \end{cases}$$

Note that each vertex in the bubble B_{ux_i} is adjacent to the vertex u and all vertices in the bubble B_{ux_j}, for each $i, j = 1, 2, 3, ..., k$. Moreover, when m is odd, each vertex in the crust C_u is adjacent to all vertices in the bubble B_{ux_j} for each $j = 1, 2, 3, ..., k$. Thus, a complete subgraph of order $\omega\left(W_k^{\frac{m}{n}}\right)$ is formed and is the maximum clique.

Each vertex in the maximum clique has to be colored with different colors since they are adjacent to one another. So, the number of colors used so far is $\omega\left(W_k^{\frac{m}{n}}\right)$. Next, we will color the vertices that belong to other bubbles and crusts without adding new colors.

Color every vertex x_i using the color of vertex u. Since the vertices in B_{x_iu} may be adjacent to some vertices in B_{ux_i}, we color the vertices in B_{x_iu} with the color of the vertices in $B_{ux_{i+1}}$ with attention to their ordering. Color all vertices in the bubble $B_{x_ix_{i+1}}$ using the same colors and sequence as the vertices in B_{ux_i}. Color all vertices in $B_{x_ix_{i-1}}$ using the same colors and sequence as the vertices in $B_{ux_{i+2}}$. Thus, all bubbles have been colored. This method of coloring the bubbles is described by the following algorithm.

Algorithm 2 Coloring the Bubble of x_i

```
1: for i = 1, 2, ..., n do
2:     for j = 1, 2, ..., ⌊m/2⌋ do
3:         x_i^{n-j} ← x_{i+1}^j
4:         x_{i,i+1}^j ← x_i^j
5:         x_{i,i-1}^j ← x_{i+2}^j
6:     end for
7: end for
```

Note that the first vertex in the middle part M_{ux_i} can be colored by the color of the last vertex in the bubble $B_{ux_{i+2}}$. Color the last vertex in the middle part M_{ux_i} (or, when m is odd, the vertices in the crust C_{x_i} that do not belong to the hyperedge ux_{i-1}) using the color of the last vertex in the bubble $B_{ux_{i-1}}$. Continue this way of coloring to color the vertices in the middle part M_{ux_i} from the front and back using the colors of the vertices in $B_{ux_{i+2}}$ and $B_{ux_{i-1}}$ alternatingly, with attention to its order, until all vertices in the middle part has been colored or until there are no colors left. When the number of middle part is less than m and m is odd, it is possible that one vertex in the middle part remains uncolored. That vertex is x_i^{m+1} and so can be colored using the color of the vertex u.

Using the same method, we can color the vertices in the middle part $M_{x_i x_{i+1}}$. Color the first vertex in $M_{x_i x_{i+1}}$ with the color of the last vertex in $B_{x_i u}$ and color the last vertex in $M_{x_i x_{i+1}}$ (or, when m is odd, the vertex in the crust $C_{x_{i+1}}$ that belongs to the hyperedge $x_i x_{i+1}$) with the color of the last vertex in $B_{x_i x_{i-1}}$. Continue this way of coloring to color the vertices in $M_{x_i x_{i+1}}$ alternately from the front and back using the colors of the vertices from the bubble $B_{x_i u}$ for the front end of $M_{x_i x_{i+1}}$ and the bubble $B_{x_i x_{i-1}}$ for the back end of $M_{x_i x_{i+1}}$. If $x_{i,i+1}^{m+1}$ has not yet been colored, color it with the color of x_i.

Using the explained method above, if the number of vertices in the middle part is less than m, then all vertices in the middle parts have already been colored, resulting in a vertex coloring for $W_k^{\frac{m}{n}}$ with $\omega(W_k^{\frac{m}{n}})$ colors. This method of coloring can be described by the following algorithm. We denote the number of vertices in a middle part as Λ.

Algorithm 3 Coloring the Middle Part for Even m

$p \leftarrow 0, j \leftarrow 1$
for $i = 1, 2, \dots k$ **do**
    while $p \neq \Lambda$ **do**
        $x_i^{\lfloor \frac{m}{2} \rfloor + j} \leftarrow x_{i+2}^{\lfloor \frac{m}{2} \rfloor + 1 - j}$
        $x_{i,i+1}^{\lfloor \frac{m}{2} \rfloor + j} \leftarrow x_i^{n - \lfloor \frac{m}{2} \rfloor - 1 + j}$
        if $\lfloor \frac{m}{2} \rfloor + j \neq \lfloor \frac{m}{2} \rfloor + \Lambda + 1 - j$ **then**
            $x_i^{\lfloor \frac{m}{2} \rfloor + \Lambda + 1 - j} \leftarrow x_{i-1}^{\lfloor \frac{m}{2} \rfloor + 1 - j}$
            $x_{i,i+1}^{\lfloor \frac{m}{2} \rfloor + \Lambda + 1 - j} \leftarrow x_{i,i-1}^{\lfloor \frac{m}{2} \rfloor + 1 - j}$
            $p \leftarrow p + 2$
            $j \leftarrow j + 1$
        else
            $p \leftarrow p + 1$
            $j \leftarrow j + 1$
        end if
    end while
end for

Algorithm 4 Coloring the Middle Part for Odd m

$p \leftarrow 0, j \leftarrow 1$
for $i = 1, 2, \ldots, k$ **do**
  while $p \neq \Lambda + 1$ **do**
    $x_i^{\lfloor \frac{m}{2} \rfloor + j + 1} \leftarrow x_{i+2}^{\lfloor \frac{m}{2} \rfloor + 1 - j}$
    $x_{i,i+1}^{\lfloor \frac{m}{2} \rfloor + j + 1} \leftarrow x_i^{n - \lfloor \frac{m}{2} \rfloor - 1 + j}$
    if $\lfloor \frac{m}{2} \rfloor + j + 1 \neq \lfloor \frac{m}{2} \rfloor + \Lambda + 3 - j$ **then**
      $x_i^{\lfloor \frac{m}{2} \rfloor + \Lambda + 3 - j} \leftarrow x_{i-1k}^{\lfloor \frac{m}{2} \rfloor + 1 - j}$
      $x_{i,i+1}^{\lfloor \frac{m}{2} \rfloor + \Lambda + 3 - j} \leftarrow x_{i,i-1}^{\lfloor \frac{m}{2} \rfloor + 1 - j}$
      $p \leftarrow p + 2$
      $j \leftarrow j + 1$
    else
      $p \leftarrow p + 1$
      $j \leftarrow j + 1$
    end if
  end while

Algorithm 4 Coloring the Middle Part for Odd m

  $x_{i,i+1}^{\lfloor \frac{m}{2} \rfloor + 1} \leftarrow x_i^{\lfloor \frac{m}{2} \rfloor + \Lambda + 2}$
end for

If the number of vertices in the middle part exceeds or equal to m, then there are vertices in the middle part that have not yet been colored. Consider the middle part M_{ux_i}. The vertex x_i^{m+1} can be colored using the color of the vertex u. For the remaining vertices, the vertex x_i^{m+p+1} can be colored using the same color as vertex x_i^p. In this case, coloring all of the remaining vertices in this manner, all vertices in the middle parts M_{ux_i} have been colored.

Consider the middle part $M_{x_i x_{i+1}}$. The vertex $x_{i,i+1}^{m+1}$ can be colored using the color of the vertex x_i. For the remaining vertices, the vertex $x_{i,i+1}^{m+p+1}$ can be colored using the color of the vertex $x_{i,i+1}^{p}$. Therefore, all vertices in the middle parts $M_{x_i x_{i+1}}$ have been colored, resulting in a coloring for all vertices in $W_k^{\frac{m}{n}}$ with $\omega\left(W_k^{\frac{m}{n}}\right)$ colors.

This method for coloring the remaining vertices in the middle part can be described by the following algorithm.

Algorithm 5 Coloring the Rest of the Middle Part for Even m

$p \leftarrow m-1$, $j \leftarrow \left\lfloor \frac{m}{2} \right\rfloor + 1$, $\alpha \leftarrow 0$, $\beta \leftarrow 0$
for $i = 1.2, \ldots, k$ **do**
  while $p \neq \Lambda$ **do**
    if $\beta \equiv 0 \pmod 2$ **then**
      $x_i^{\left\lfloor \frac{m}{2} \right\rfloor + j} \leftarrow x_i^{\alpha}$
      $x_{i,i+1k}^{\left\lfloor \frac{m}{2} \right\rfloor + j} \leftarrow x_{i,i+1}^{\alpha}$
    else
      $x_i^{\left\lfloor \frac{m}{2} \right\rfloor + j} \leftarrow x_{i+2}^{\left\lfloor \frac{m}{2} \right\rfloor - \alpha}$
      $x_{i,i+1}^{\left\lfloor \frac{m}{2} \right\rfloor + j} \leftarrow x_i^{n - \left\lfloor \frac{m}{2} \right\rfloor + \alpha}$
    end if
    $\alpha \leftarrow \alpha + 1$
    if $\alpha = \left\lfloor \frac{m}{2} \right\rfloor + 1$ or ($\alpha = \left\lfloor \frac{m}{2} \right\rfloor$ and $\beta \equiv 1 \pmod 2$)) **then**
      $\alpha \leftarrow 0$
      $\beta \leftarrow \beta + 1$
    end if
    $p \leftarrow p + 1$
    $j \leftarrow j + 1$
  end while
end for

Algorithm 6 Coloring the Rest of the Middle Part for Odd m

$p \leftarrow m$, $j \leftarrow \left\lfloor \frac{m}{2} \right\rfloor + 2$, $\alpha \leftarrow 1$, $\beta \leftarrow 0$
for $i = 1, 2, \ldots, k$ **do**
    while $p \neq \Lambda$ **do**
        if $\beta \equiv 0 \pmod 2$ **then**
            $x_i^{\left\lfloor \frac{m}{2} \right\rfloor + j} \leftarrow x_i^{\alpha}$
            $x_{i,i+1}^{\left\lfloor \frac{m}{2} \right\rfloor + j} \leftarrow x_{i,i+1}^{\alpha}$
        else
            $x_i^{\left\lfloor \frac{m}{2} \right\rfloor + j} \leftarrow x_{i+2}^{\left\lfloor \frac{m}{2} \right\rfloor - \alpha}$
            $x_{i,i+1}^{\left\lfloor \frac{m}{2} \right\rfloor + j} \leftarrow x_i^{n - \left\lfloor \frac{m}{2} \right\rfloor + \alpha}$
        end if
        $\alpha \leftarrow \alpha + 1$
        if $\alpha = \left\lfloor \frac{m}{2} \right\rfloor + 2$ or ($\alpha = \left\lfloor \frac{m}{2} \right\rfloor$ and $\beta \equiv 1 \pmod 2$) **then**
            $\alpha \leftarrow 0$
            $\beta \leftarrow \beta + 1$
        end if
        $p \leftarrow p + 1$
        $j \leftarrow j + 1$
    end while
end for

To illustrate, the result of this whole process applied to $W_4^{\frac{2}{6}}$ is given in Figure 4. Note that the edges generated by the 2-power operation are omitted for readability. Firstly, color the bubbles of the graph using Algorithm 2. Since $m = 2$ is even, Algorithm 3 is then used to color a portion of the middle parts. Finally, the remaining middle parts are colored using Algorithm 5.

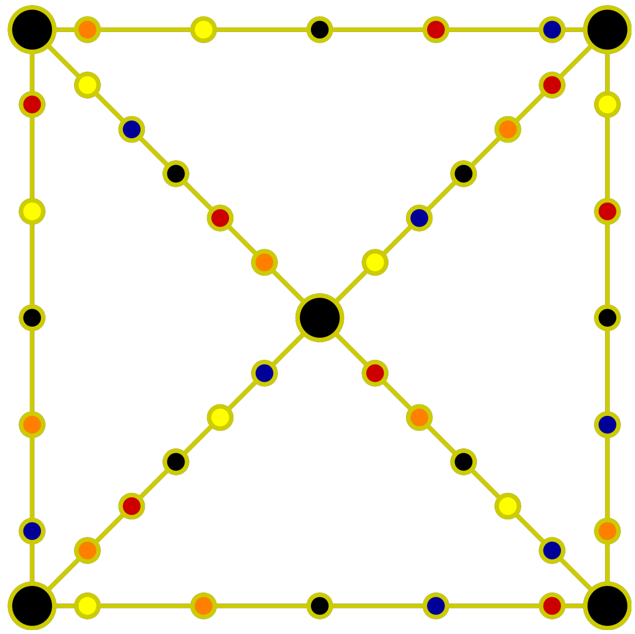

Figure 4: Example of $W_4^{\frac{2}{6}}$

We conclude that for all $k \geq 3$, the wheel graph W_k on $k+1$ vertices fullfils the conjecture, that is, $\chi\left(W_k^{\frac{m}{n}}\right) = \omega\left(W_k^{\frac{m}{n}}\right)$. □

Theorem 3.2 implies these corollaries.

Notice that after coloring every vertex in $W_k^{\frac{m}{n}}$, where k is even, we can alternately remove the hyperedges of the rim, resulting in a coloring for the friendship graph $F_{\frac{k}{2}}^{\frac{m}{n}}$ with the same parameters m and n. Therefore, the friendship graph also satisfies the conjecture.

Corollary 3.1. *If F_k is a friendship graph with $k \geq 2$, then $\chi\left((F_k)^{\frac{m}{n}}\right) = \omega\left((F_k)^{\frac{m}{n}}\right)$.*

Notice that after coloring every vertex in $W_k^{\frac{m}{n}}$, we can remove one hyperedge of the rim, resulting in a coloring for the fan graph $F_{1,k}^{\frac{m}{n}}$ with the same parameters m and n. Therefore, the fan graph also satisfies the conjecture.

Corollary 3.2. *If $F_{1,k}$ is a fan graph with $k \geq 2$, then $\chi\left((F_{1,k})^{\frac{m}{n}}\right) = \omega\left((F_{1,k})^{\frac{m}{n}}\right)$.*

4. Conclusion

In this research, we have proven the conjecture for some classes of graphs, that is, star, wheel, fan, and friendship graphs, each with $n \geq 3$ vertices. For future studies, we recommend extending this analysis to other classes of graphs to further validate or challenge the conjecture across different structures. Additionally, searching for further counterexamples to the conjecture may prove to be valuable, possibly further uncovering the conditions such that the conjecture does not hold.

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