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The number of spanning trees of cyclic snakes

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Abstract

A cyclic snake is a connected graph formed by connecting, by means of vertex amalgamation, a certain number of copies of the cycle C_n, in such a way that the ith copy of C_n is connected with the $(i+1)$th copy, resulting in a graph with maximum degree 4. Spanning trees of this type of graph can be easily found, but finding the number of nonisomorphic spanning trees of a given cyclic snake is a more challenging problem. In this work, we investigate the number of cyclic snakes formed with k copies of C_n, the number of spanning trees of any given cyclic snake. We also classified these trees according to their diameters. Finally, we study the morphology of the trees associated to the snakes where the distance between cut-vertices is a constant.

Keywords: cyclic snake, spanning tree
Mathematics Subject Classification : 05C30, 05C75, 05C05

1. Introduction: Cyclic Snakes

A kC_n-*snake* is a connected graph with k blocks whose block-cutpoint graph is a path and each the blocks is isomorphic to the cycle C_n. This definition, introduced in [1] is a generalization of the concepts of triangular cacti and triangular snake introduced by Rosa ([4]) and Moulton ([3]), respectively. Note that if k is either 1 or 2, there exists only one kC_n-snake; for $n \geq 3$ the situation is more complex. Assume $k \geq 2$ and let $B_1, B_2, \ldots, B_k$ be copies of C_n, a kC_n-snake is formed by amalgamating a vertex of B_i with a vertex of B_{i+1}, for each $1 \leq i \leq k-1$, in such a way that the resulting graph is connected of order $k(n-1)+1$ and size kn; this graph has $k(n-2)+2$ vertices of degree 2 and $k-1$ vertices of degree 4. We must also notice that, the vertex connectivity, κ, of any kC_n-snake is 2 when $k=1$ and 1 for $k \geq 2$. In Figure 1 we show all six nonisomorphic $4C_6$-snakes.

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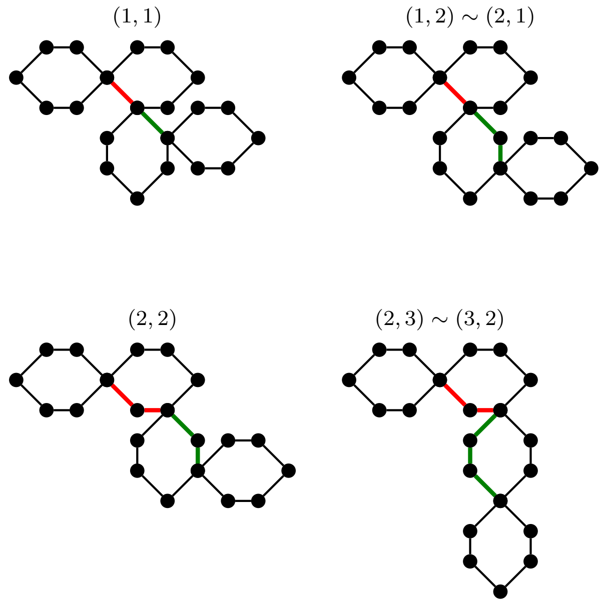

Figure 1. All $4C_6$-snakes with their characteristic strings

The case of triangular snakes is unique because for each $k \geq 1$, there is only one kC_3-snake. The same occurs when k is either 1 or 2. In all other combinations of k and n, there are more than one cyclic snake. This fact motivates the search of system that allows us to distinguish between any two kC_n-snakes. Suppose $k \geq 3$ and let Γ be a kC_n-snake formed with the blocks $B_1, B_2, \ldots, B_k$. For each $1 \leq i \leq k-1$, let v_i denote the vertex shared by B_i and B_{i+1}, in other terms, $v_1, v_2, \ldots, v_{k-1}$ are the cut-vertices of Γ. If $d_i = \text{dist}(v_i, v_{i+1})$, then the string $(d_1, d_2, \ldots, d_{k-2})$ can be used to identify the cyclic snake Γ. Thus, we have a surjective function between the set $\mathcal{S}(n, k)$ of all strings of length $k-2$ whose entries are in $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, and the set of all kC_n-snakes. Furthermore, let $S_1, S_2 \in \mathcal{S}(n, k)$, we say that S_1 and S_2 are *equivalent* if either $S_1 = S_2$ or S_2 is the reverse of S_1. In this way, we have a bijective function between the set of equivalence classes induced on $\mathcal{S}(n, k)$ and the set of all kC_n-snakes. We say that $S = (d_1, d_2, \ldots, d_{k-2})$ is the *characteristc string* of Γ if $d_i = \text{dist}(v_i, v_{i+1})$ for each $1 \leq i \leq k-1$. We say that B_1 and B_k are the *external* blocks of Γ and $B_2, B_3, \ldots, B_{k-1}$ are the *internal* blocks. Note that each internal block of Γ has exactly two of the cut-vertices while each external block has only one cut-vertex. In Figure 1 we associated each $4C_6$-snake with its corresponding characteristic string(s); in addition, we colored the subpath of length d_1 and d_2. Thus, for example, the graph on the top right corner can be referred to as Γ with characteristic string $(1, 3)$.

We can use these characteristic strings to determine the number of nonisomorphic cyclic snakes formed by amalgamating k copies of the cycle C_n. Before doing that, we must observe that when $n > 3$ is odd, then in any kC_n-snake, where $k \geq 3$, the distance between the cut-vertices of any internal block is at most $\lfloor n/2 \rfloor$. This implies that for any even value of n, the sets $\mathcal{S}(n, k)$ and $\mathcal{S}(n+1, k)$ are exactly the same. Moreover, $\mathcal{S}(m, k) \subseteq \mathcal{S}(n, k)$ for each $m \leq n$. Consequently, in the rest of this section we assume that n is even.

Let $n \geq 4$. The set $\mathcal{S}(n,k)$ has cardinality $\lfloor n/2 \rfloor^{k-2}$. Since the equivalence classes in $\mathcal{S}(n,k)$ have one or two members, we want to determine the number of equivalence classes with only one member; once this number has been determined, we add it to $\lfloor n/2 \rfloor^{k-2}$, and divide this sum by 2. This final quotient gives us the number of equivalence classes in $\mathcal{S}(n,k)$, that is, the number of nonisomorphic cyclic snakes with k blocks isomorphic to C_n.

A string $S = (d_1, d_2, \ldots, d_{k-2})$ is said to be *reversible* if it is equal to its reverse, i.e., for each $1 \leq i \leq k-2$, we have $d_i = d_{k-1-i}$. We are interested now in the number of reversible strings in $\mathcal{S}(n,k)$. Let $1 \leq i \leq \lceil \frac{k-2}{2} \rceil$, the entry d_i of S can take any value in $\{1, 2, \ldots, \frac{n}{2}\}$, but once the value of d_i has been determined, the value of d_{k-1-i} is unique because in a reversible string both entries must have the same value. Therefore, the number of reversible strings in $\mathcal{S}(n,k)$ is

$$r(n,k) = \left(\frac{n}{2}\right)^{\lceil \frac{k-2}{2} \rceil}.$$

With this result we can prove the following result about the number of nonisomorphic kC_n-snakes.

Theorem 1.1. *For each even value of $n \geq 4$, and for every positive value of k, the number cs(n,k) of nonisomorphic kC_n-snakes is:*

(i) 1 *if* $k = 1, 2$,

(ii) $\frac{1}{2}\left(\left(\frac{n}{2}\right)^{k-2} + \left(\frac{n}{2}\right)^{\lceil \frac{k-2}{2} \rceil}\right)$ *otherwise.*

Proof. If $k = 1$, the graph $1C_n$-snake is the cycle C_n, therefore $\mathrm{cs}(n,1) = 1$. When $k = 2$, the graph $2C_n$-snake is the one-point union (or vertex amalgamation) of two copies of C_n, and $\mathrm{cs}(n,2) = 1$. Suppose now that $k \geq 3$. There exists $(n/2)^{k-2}$ strings of length $k-2$ whose entries are in $\{1, 2, \ldots, n/2\}$. Among these strings, exactly $(n/2)^{\lceil \frac{k-2}{2} \rceil}$ are reversible. If A is the set of all strings of length $k-2$ with entries in $\{1, 2, \ldots, n/2\}$ and B is the subset of A formed by all reversible strings, then $A \cup B$ is a multiset of cardinality

$$\left(\frac{n}{2}\right)^{k-2} + \left(\frac{n}{2}\right)^{\lceil \frac{k-2}{2} \rceil}.$$

Now, each kC_n-snake is associated with exactly two strings in $A \cup B$. Consequently, the number of nonisomorphic kC_n-snakes is given by

$$\frac{1}{2}\left(\left(\frac{n}{2}\right)^{k-2} + \left(\frac{n}{2}\right)^{\lceil \frac{k-2}{2} \rceil}\right).$$

$\square$

As we mentioned before, when $k \geq 3$ and n is even, the distance between the cut-vertices of any internal block in a kC_n-snake is $\lfloor \frac{n+1}{2} \rfloor$, that is $\frac{n}{2}$. Then, the number of nonisomorphic kC_{k+1}-snakes is also $\mathrm{cs}(n,k)$, for each even value of $n \geq 4$.

Corollary 1.1. *If $n+1 \geq 5$ is odd, the number of nonisomorphic kC_{k+1}-snakes is cs(n, k).*

Remark 1.1. When $n+1=3$, we have that $\frac{n}{2}=1$ and $\text{cs}(2,k)=\frac{1}{2}\left(1^{k-2}+1^{\lceil\frac{k-2}{2}\rceil}\right)=1$, which is completely consisting with the fact that there is only one kC_3-snake for each $k \geq 1$.

In Table 1 we present the initial values of $\text{cs}(n,k)$ for each $4 \leq n \leq 18$ and $1 \leq k \leq 10$.

Table 1. Number of nonisomorphic kC_n-snakes, $n \geq 4$ even

$n \mid k$	1	2	3	4	5	6	7	8	9	10
4	1	1	2	3	6	10	20	36	72	136
6	1	1	3	6	18	45	135	378	1134	3321
8	1	1	4	10	40	136	544	2080	8320	32,896
10	1	1	5	15	75	325	1625	7875	39,375	195,625
12	1	1	6	21	126	666	3996	23,436	140,616	840,456
14	1	1	7	28	196	1225	8575	58,996	412,972	2,883601
16	1	1	8	36	288	2080	16,640	131,328	1,050,624	8,390,656
18	1	1	9	45	405	3321	29,889	266,085	2,394,756	21,526,641
20	1	1	10	55	550	5050	30,500	500,500	5,005,000	50,005,000
22	1	1	11	66	726	7381	81,191	886,446	9,750,906	107,186,761

2. The Number of Spanning Trees

Let G be a connected graph. A subgraph H of G is a *spanning subgraph* if all its vertices are also vertices of G. A *spanning tree* of G is a spanning subgraph of G that is a tree. Spanning trees has been studied extensively, many of the results are collected in the book of Wu and Chao [5]. One of the lines of investigation in this context is the enumeration problem, that is, determining the number of spanning trees of a given graph.

In this section we want to determine the number, st(Γ), of spanning trees of Γ, where Γ is the kC_n-snake with characteristic string $S=(d_1,d_2,\ldots,d_{k-2})$. As mentioned before, when $k=1$, there exists a unique cyclic snake that is the cycle C_n itself. Since the path P_n is the unique spanning tree of C_n, we have st$(\Gamma \cong C_n)=1$. Suppose now that $k \geq 2$. Since a spanning tree of Γ is obtained by deleting exactly one edge from each cycle, we are interested in the number of ways this deletion process can be done. Because a block of Γ can be internal or external, we analyze each type independently.

Let Γ be a kC_n-snake whose blocks are $B_1,B_2,\ldots,B_k$; let $v_1,v_2,\ldots,v_{k-1}$ be the cut-vertices of Γ, where v_i is the vertex connecting B_i and B_{i+1}, let $e_i=\alpha_i\omega_i$ be an edge of B_i such that dist$(v_i,\,\alpha_i) <$ dist$(v_i,\,\omega_i)$ in the path B_i-e_i. If B_i is an external block, then B_i contains only one cut-vertex of Γ, the vertex v_i; we associate the path B_i-e_i with the pair $(x_1,\,x_2)$, where $x_1=$ dist$(v_i,\,\alpha_i)$ and $x_2=$ dist$(v_i,\,\omega_i)$. Thus,

$$x_1+x_2=n-1 \tag{1}$$

where x_i is a nonnegative integer, because it is possible that $v_i=\alpha_i$. This equation has n solutions, however if $(a,\,b)$ is a solution, so it is $(b,\,a)$. For our counting purposes, both solutions are considered the same. Then, we say that equation (1) has $\lceil n/2 \rceil$ distinct solutions. We refer to these solutions as 2-*strings*.

If B_i is an internal block, then B_i contains two cut-vertices of Γ, v_{i-1} and v_i; in this case, we associate the path $B_i - e_i$ with the ordered triple $(x_1,\, x_2,\, x_3)$, where $x_1 = \text{dist}(v_{i-1},\, \alpha_i)$, $x_2 = \text{dist}(v_{i-1},\, v_i)$, and $x_3 = \text{dist}(v_i,\, \omega_i)$. Thus,

$$x_1 + x_2 + x_3 = n - 1. \tag{2}$$

Since it is possible that $v_{i-1} = \alpha_i$ and/or $v_i = \omega_i$, we get that x_1 and x_3 are nonnegative integers, the fact that Γ is a cyclic snake implies that x_2 is a positive integer.

Equation (2) is equivalent to

$$x_1 + x_2' + x_3 = n - 2, \tag{3}$$

where $x_2' = x_2 - 1$. Now the variables x_1, x_2', and x_3 represent nonnegative integers. The solutions of both equations are in a one-to-one correspondence. The number of nonnegative integer solutions of equation (3) is

$$\binom{n-2+(3-1)}{n-2} = \binom{n}{n-2} = \frac{n(n-1)}{2}.$$

Therefore, there are $\frac{n(n-1)}{2}$ solutions of equation (2). Since x_1 and x_3 are defined by the position of e_i with respect to the cut-vertices of B_i, a solution $(a,\, b,\, c)$ of (2) is different than the solution $(c,\, b,\, a)$, unless $a = c$. We refer to the solutions of equation (2) as 3-*strings*.

The spanning tree T of Γ, obtained by deleting the edges $e_1, e_2, \ldots, e_k$ can be represented by the string of length $2 \cdot 2 + 3(k-2)$ formed by the 2-strings associated to the two external blocks, and the 3-strings associated to the $k-2$ internal blocks. In order to avoid confusion, we refer to this string as the *long string* of Γ. In Figure 4 we show two (nonisomorphic) spanning trees of the $6C_8$-snake with characteristic string $(2,\, 3,\, 4,\, 1)$. The tree T_1 is associated with the long string $(4,\, 3), (3,\, 2,\, 2), (2,\, 3,\, 2), (1,\, 4,\, 2), (2,\, 1,\, 4), (1,\, 6)$. The tree T_2 is associated with the long string $(5,\, 2), (0,\, 6,\, 1), (1,\, 5,\, 1), (2,\, 4,\, 1), (0,\, 7,\, 0), (0,\, 7)$.

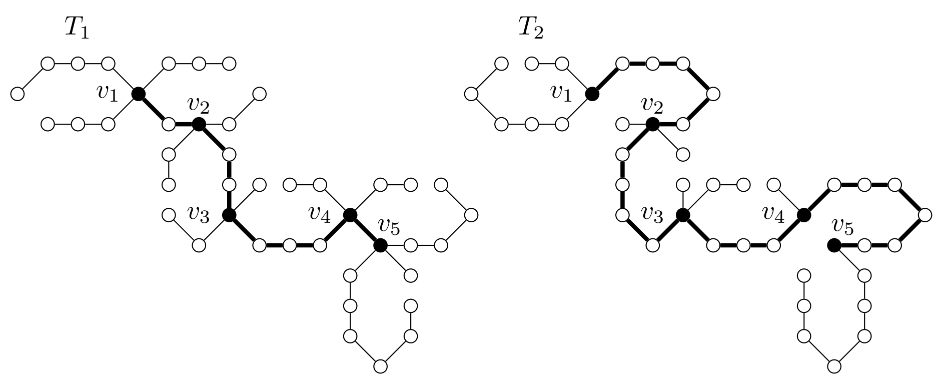

Figure 2. Two spanning trees of the $6C_8$-snake with characteristic string $(2,\, 3,\, 4,\, 1)$

In general, for a cyclic snake Γ, the long string looks like

$$(x_1^1,\, x_2^1), (x_1^2,\, x_2^2,\, x_3^2), (x_1^3,\, x_2^3,\, x_3^3), \ldots, (x_1^{k-1},\, x_2^{k-1},\, x_3^{k-1}), (x_1^k,\, x_2^k),$$

where the supra index indicates the block of Γ corresponding to that particular substring. When analyzing equation (1), we mentioned that the solutions (x_1, x_2) and (x_2, x_3) are considered the same. Thus, without loss of generality, we assume that $x_1^1 \geq x_2^1$ and $x_1^k \leq x_2^k$. Note that the spanning tree can be described by the reverse string, i.e.,

$$(x_2^k, x_1^k), (x_3^{k-1}, x_2^{k-1}, x_1^{k-1}), (x_3^{k-2}, x_2^{k-2}, x_1^{k-2}), \ldots, (x_3^2, x_2^2, x_1^2), (x_2^1, x_1^1),$$

which is obtained by using B_k instead of B_1 as the head of the snake. This tells us that if a cyclic snake is "asymmetric", in the sense that its characteristic string $(d_1, d_2, \ldots, d_{k-2})$ is not reversible, there are two distinct long strings associated with any of its spanning trees, and if the characteristic string is reversible and the spanning tree is achiral, there is only one long string associated with the tree.

Recall that $d_i = \text{dist}(v_i, v_{i+1})$, where v_i and v_{i+1} are the cut vertices of Γ contained in B_{i+1}. Thus, each internal block of Γ is associated with an entry d in the characteristic string of Γ . So, how many of the $n(n-1)/2 = \binom{n}{2}$ 3-strings correspond to the specific value of d? To answer this question we concentrate on the nature of these solutions. We begin this inspection by considering the case where n is odd. Suppose that the 3-string (x_1, x_2, x_3) is a solution of equation (2). Since x_2 can be either d or $n-d$ and $d \in \{1, 2, \ldots, \frac{n-1}{2}\}$, the $\binom{n}{2}$ solutions of (2) can be classified into $\frac{n-1}{2}$ sets, where the elements of each set are determined by the $\frac{n-1}{2}$ possible values of d. That is, for a fixed value of $d \in \{1, 2, \ldots, \frac{n-1}{2}\}$, all 3-strings of the forms (x_1, d, x_3) and $(x_1', n-d, x_3')$ are in the same set. In our enumeration process is essential the cardinality of each of these sets. For the first type of 3-strings we have

$$x_1 + d + x_3 = n - 1 \quad \Leftrightarrow \quad x_1 + x_3 = n - 1 - d. \tag{4}$$

The number of nonnegative integer solutions of equation (4) is

$$\binom{n-1-d+(2-1)}{n-1-d} = \binom{n-d}{n-d-1} = n - d.$$

Similarly, for the second type of 3-strings we get

$$x_1' + n - d + x_3' = n - 1 \quad \Leftrightarrow \quad x_1' + x_3' = d - 1. \tag{5}$$

The number of nonnegative integer solutions of this equation is

$$\binom{d-1+(2-1)}{d-1} = \binom{d}{d-1} = d.$$

Therefore, each of these sets has exactly $n - d + d = n$ elements.

Another aspect of the 3-strings that we must consider is the reversibility. A 3-string (x_1, x_2, x_3) is said to be *reversible* if $x_1 = x_3$. If we impose this condition on the solutions of equations (4) and (5), we see that one of these equations has a unique reversible solution while the other equation has no reversible solution. Hence, each set of 3-strings has exactly n elements and only one of them is reversible. We show below the sets of 3-strings for the cases $n = 5$ and $n = 7$. On both cases, we have highlighted the reversible 3-strings.

$n=5$	**(0,4,0)**, (3,1,0), (0,1,3), (2,1,1), (1,1,2) (1,3,0), (0,3,1), (2,2,0), (0,2,2), **(1,2,1)**
$n=7$	**(0,6,0)**, (5,1,0), (0,1,5), (4,1,1), (1,1,4), (3,1,2), (2,1,3) (1,5,0), (0,5,1), (4,2,0), (0,2,4), (3,2,1), (1,2,3), **(2,2,2)** (2,4,0), (0,4,2), **(1,4,1)**, (3,3,0), (0,3,3), (2,3,1), (1,3,2)

Now we analyze the case n even. We proceed as in the previous case. Again, we classify the $\binom{n}{2}$ solutions of equation (2) into $\frac{n}{2}$ sets, altough now not all sets have the same cardinality, this is due to the fact that when $d=\frac{n}{2}$, $n-d$ is also equal to $\frac{n}{2}$. Hence, for each $d \in \{1, 2, \ldots, \frac{n}{2}-1\}$, the set that contains the 3-strings $(x_1,\, d,\, x_3)$ and $(x_1',\, n-d,\, x_3')$ has exactly n elements, while the set containing the strings $(x_1,\, \frac{n}{2},\, x_3)$ has cardinality $\frac{n}{2}$ because the equation

$$x_1 + \frac{n}{2} + x_3 = n-1 \quad \Leftrightarrow \quad x_1 + x_3 = \frac{n}{2} - 1 \tag{6}$$

has $\binom{n/2-1+(2-1)}{n/2-1} = \binom{n/2}{n/2-1} = \frac{n}{2}$ solutions.

When studying the reversibility of the 3-strings we need to consider two cases.

- If $n \equiv 0 (\text{mod } 4)$, then the equation (4) becomes

  $$2x_1 = n-1-d,$$

  which has solution if and only if d is odd. Similarly, equation (5) changes to

  $$2x_1' = d-1$$

  that has solutions only when d is odd. Therefore, for each odd value of d, the set containing the 3-strings $(x_1,\, d,\, x_3)$ and $(x_1',\, n-d,\, x_3')$ has exactly two reversible strings, and if d is even, there are no reversible 3-strings.

- If $n \equiv 2 (\text{mod } 4)$, the situation is basically the same with the exception of the case $d=\frac{n}{2}$, where d is odd; in this case the set containing the 3-strings of the form $(x_1,\, \frac{n}{2},\, x_3)$ has exactly one reversible element.

We show below the sets of 3-strings for the cases $n=6$ and $n=8$. As we did before, we highlight the reversible 3-strings.

$n=6$	**(0,5,0)**, (4,1,0), (0,1,4), (3,1,1), (1,1,3), **(2,1,2)** (1,4,0), (0,4,1), (3,2,0), (0,2,3), (2,2,1), (1,2,2) (2,3,0), (0,3,2), **(1,3,1)**
$n=8$	**(0,7,0)**, (6,1,0), (0,1,6), (5,1,1), (1,1,5), (4,1,2), (2,1,4), **(3,1,3)** (1,6,0), (0,6,1), (5,2,0), (0,2,5), (4,2,1), (1,2,4), (3,2,2), (2,2,3) (2,5,0), (0,5,2), **(1,5,1)**, (4,3,0), (0,3,4), (3,3,1), (1,3,3), **(2,3,2)** (3,4,0), (0,4,3), (2,4,1), (1,4,2)

Theorem 2.1. *Let Γ be the kC_n-snake with characteristic string $S=(d_1, d_2, \ldots, d_{k-2})$ with r entries equal to $\frac{n}{2}$. The number st(Γ) of nonisomorphic spanning trees of Γ is,*

1. *when* $k = 1$, *st*$(\Gamma) = 1$,
2. *when* n *is odd,*
$$st(\Gamma) = \frac{1}{2}\Big[\Big(\frac{n+1}{2}\Big)^2 n^{k-2} + \Big(\frac{n+1}{2}\Big) n^{\lfloor \frac{k-2}{2} \rfloor}\Big],$$
3. *when* n *is even*
$$st(\Gamma) = \frac{n^k}{2^{r+3}} + \frac{n^{\lfloor \frac{k}{2} \rfloor}}{2^{\theta(r)}},$$
where
$$\theta(r) = \begin{cases} (r+4)/2 & \textit{if } k \textit{ is even,} \\ (r+3)/2 & \textit{if } k \textit{ and } r \textit{ are odd,} \\ (r+2)/2 & \textit{if } k \textit{ is odd and } r \textit{ is even.} \end{cases}$$

Proof. If $k = 1$, then $\Gamma = C_n$, and P_n is the unique spanning tree of C_n. Thus, $\mathrm{st}(\Gamma) = 1$ for all $n \geq 3$.

Suppose that n is odd. Then, the number of 2-strings is $\frac{n+1}{2}$ and the number of is n 3-strings for each value of d_i; moreover, only one of the 3-strings associated to d_i is reversible. Thus, the number of long strings is
$$\Big(\frac{n+1}{2}\Big)^2 \cdot n^{k-2}.$$
When k is even, the number of reversible strings is
$$\Big(\frac{n+1}{2}\Big) \cdot n^{(k-2)/2}.$$
When k is odd, the number of reversible strings is
$$\Big(\frac{n+1}{2}\Big) \cdot n^{(k-3)/2} \cdot 1 = \Big(\frac{n+1}{2}\Big) \cdot n^{(k-3)/2}.$$
Since $\frac{k-3}{2} = \lfloor \frac{k-2}{2} \rfloor$ when k is odd and $\frac{k-2}{2} = \lfloor \frac{k-2}{2} \rfloor$ when k is even, we get
$$\mathrm{st}(\Gamma) = \frac{1}{2}\Big[\Big(\frac{n+1}{2}\Big)^2 n^{k-2} + \Big(\frac{n+1}{2}\Big) n^{\lfloor \frac{k-2}{2} \rfloor}\Big].$$

Suppose now n is even. The number of 2-strings is $\frac{n}{2}$; for each $d_i \in \{1, 2, \ldots, \frac{n-2}{2}\}$ the number of 3-strings is n, and for $d_i = \frac{n}{2}$ the number of 3-strings is 1. The number of long strings is
$$\Big(\frac{n}{2}\Big)^2 \cdot \Big(\frac{n}{2}\Big)^r \cdot n^{k-2-r} = \frac{n^k}{2^{r+2}}.$$

In order to determine the number of reversible long strings we must consider the parity of k, in addition, recall that when n is even, a 3-string is reversible only when its central entry is odd.

When k is even, any reversible long string must be associated with a characteristic string with an even value for r. Thus, the number of reversible long strings is
$$\Big(\frac{n}{2}\Big) \cdot \Big(\frac{n}{2}\Big)^{r/2} \cdot n^{(k-2-r)/2} = \frac{n^{k/2}}{2^{(r+2)/2}}$$

and

$$\mathrm{st}(\Gamma) = \frac{1}{2}\left[\frac{n^k}{2^{r+2}} + \frac{n^{k/2}}{2^{(r+2)/2}}\right] = \frac{n^k}{2^{r+3}} + \frac{n^{k/2}}{2^{(r+4)/2}}.$$

When k is odd, r can be either odd or even. Assume first that r is odd. Under this assumption, the central 3-string of a reversible long string should be reversible, consequently, the middle entry of such a 3-string must be odd, this only occurs when $n \equiv 2 (\mathrm{mod}\ 4)$ and there is only one reversible 3-string. Then, the number of reversible long strings is

$$\left(\frac{n}{2}\right) \cdot \left(\frac{n}{2}\right)^{(r-1)/2} \cdot n^{(k-3-(r-1))/2} \cdot 1 = \frac{n^{(k-1)/2}}{2^{(r+1)/2}}.$$

Therefore,

$$\mathrm{st}(\Gamma) = \frac{1}{2}\left[\frac{n^k}{2^{r+2}} + \frac{n^{(k-1)/2}}{2^{(r+1)/2}}\right] = \frac{n^k}{2^{r+3}} + \frac{n^{(k-1)/2}}{2^{(r+3)/2}} = \frac{n^k}{2^{r+3}} + \frac{n^{\lfloor \frac{k}{2} \rfloor}}{2^{\frac{r+3}{2}}}$$

because $\frac{k-1}{2} = \lfloor \frac{k}{2} \rfloor$ when k is odd.

Finally, assume that r is even. So, the central 3-string of a reversible long string corresponds to an odd value of $d_{\frac{k+1}{2}} \neq \frac{n}{2}$. There are two such 3-strings, then the number of reversible long strings is

$$\left(\frac{n}{2}\right) \cdot \left(\frac{n}{2}\right)^{r/2} \cdot n^{(k-3-r)/2} \cdot 2 = \frac{n^{(k-1)/2}}{2^{r/2}}$$

and

$$\mathrm{st}(\Gamma) = \frac{1}{2}\left[\frac{n^k}{2^{r+2}} + \frac{n^{(k-1)/2}}{2^{r/2}}\right] = \frac{n^k}{2^{r+3}} + \frac{n^{(k-1)/2}}{2^{(r+2)/2}} = \frac{n^k}{2^{r+3}} + \frac{n^{\lfloor \frac{k}{2} \rfloor}}{2^{\frac{r+2}{2}}}.$$

□

As we can see, when n is odd, the value of $\mathrm{st}(\Gamma)$ depends exclusively of the values of n and k, in other terms, it is independent of the characteristic string of Γ , which facilitates its calculation. In Table2 we present the value of $\mathrm{st}(\Gamma)$ for $n \in \{3, 5, 7, 9, 11\}$ and $1 \leq k \leq 8$.

Table 2. Number of nonisomorphic spanning trees of kC_n-snake for n odd and $1 \leq k \leq 8$

$n \mid k$	1	2	3	4	5	6	7	8	9	10
3	1	3	7	21	57	171	495	1485	4401	13,203
5	1	6	24	120	570	2850	14,100	70,500	351,750	1,758,750
7	1	10	58	406	2758	19,306	134,554	941,878	6,589,030	46,123,210
9	1	15	115	1035	9135	82,215	738,315	6,644,835	59,788,935	538,100,415
11	1	21	201	2211	23,991	263,901	2,899,281	31,892,091	350,773,071	3,858,503,781

Note that the sequence formed by the consecutive values of $\mathrm{st}(\Gamma)$ with $k = 2$ is the sequence of triangular numbers, that corresponds to A000217 in On-Line Encyclopedia of Integer Sequences (OEIS), the sequence formed when $k = 3$ is A081436 in OEIS.

The values of $\mathrm{st}(\Gamma)$ are more complicated to analyze when n is even because they depend not only on n and k but also on r. However some interesting sequences are obtained when the characteristic string of Γ has the form $\left(\frac{n}{2}, \frac{n}{2}, \ldots, \frac{n}{2}\right)$. Cyclic snakes of this kind are called *linear* in [1]. Recall that in this case, the formula for $\mathrm{st}(\Gamma)$ changes according to $\theta(r)$.

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