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A note on second degrees in graphs

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Abstract

The second degree of a node x in a graph $\Gamma = (V, E)$, denoted by $deg_2(x)$, is the number of nodes at distance two from x in a graph Γ. In the present article, we are interested in examination of the second degrees properties in a graph. The old bounds and the general formulas of the second degree of some graph operations are collected. We provide an improvement on the useful result "$deg_2(x) \leq \big(\sum_{y \in N(x)} deg(y)\big) - deg(x)$, for every $x \in V(\Gamma)$", by adding a term of the triangles number in a graph, in order to the equality holds for each quadrangle-free graph. Further, upper and lower bounds for the maximum and minimum second degrees are established. Finally the second degree-sum formula are derived. In addition, bounds on second degree-sum are also established.

Keywords: Second degree (of node), Distance in a graph, Graph Operations.
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1. Introduction

Throughout this work, we will examine the simple graphs only, such that is an undirected graph $\Gamma = (V, E)$ with neither self-loops, directed, nor multiple links and with finite numbers of nodes. We will denote by $V = V(\Gamma)$ to the node set, and $E = E(\Gamma)$ to the link set of Γ, whereas $p = |V|$ and $q = |E|$, indicate the order and size of Γ, respectively. The distance $d(x, y)$ from a node x to a node y in Γ is the number of links in the shortest path connecting them. The eccentricity $e(x)$ of x is the maximum distance between x and other nodes in Γ, the radius of Γ, denote

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$rad(\Gamma)$, is the minimum of the eccentricities in Γ, and the maximum is the diameter $diam(\Gamma)$. If $diam(\Gamma) = rad(\Gamma)$, then Γ is called a self-centered graph.

A graph with only singleton is said to be a trivial graph and denoted by K_1. The complement $\overline{\Gamma}$ of a graph Γ, is a graph with node set $V(\overline{\Gamma}) = V(\Gamma)$ and link set $E(\overline{\Gamma}) = \{\{x, y\} \subseteq V(\Gamma) : xy \notin E(\Gamma)$. If Γ isomorphic to $\overline{\Gamma}$, then Γ is called a self-complementary (in short sc-graph). The total disconnected graph is a graph with $p \geq 1$ nodes and $q = 0$ links. A k-regular graph is a graph whose all nodes have a degree equals to k. If the graph Γ have no an induced subgraph isomorphic to F, then it is called an F-free. If a graph Γ consists of k disjoint components of a graph H, then we write $\Gamma = kH$. For notation and concepts did not define here, the reader referred to books [4, 7].

The open k-distance neighborhood of a node x in a graph Γ, denoted by $N_k(x/\Gamma)$ (or $N_k(x)$ if no misunderstand), and is defined by $N_k(x) = \{y \in V(\Gamma) : d(x, y) = k\}$, and the k^{th}-degree of a node x, denote by $deg_k(x/\Gamma)$ (or $deg_k(x)$), is the number of nodes in $N_k(x)$. It is clear that, $N_1(x) = N(x)$, and $deg_1(x) = deg(x)$. are the number of nodes adjacent with x in Γ. If $k = 2$, then $N_2(x)$ and $deg_2(x)$ are depicted the second neighborhood and second degree of a node x, respectively. we will denote by $E_2(\Gamma)$ to the set of all unordered pairs of nodes of Γ which the distance between them equal two. That is $E_2(\Gamma) = \Big\{\{y, x\} \subseteq V(\Gamma) : d(x, y) = 2\Big\}$.

Topological indices (TIs) (structure-descriptor) are fixed invariants (parameters) which do not change for isomorphic graphs. These descriptors founded have many application in several field of sciences. For instance, they are very useful in the graphical construction of chemical compounds, pharmacy, genetic. A large number of topological indices (TIs) were introduced and extensively studied in an attempt to characterize the physical-chemical properties of molecules. A topological index is a function from a set af all graphs to a positive real numbers and can be defined on degree of nodes, distance between nodes or a mixed of them. Thus TIs categorized into degree-based, distance-based and degree-distance-based.

In the mathematico-chemical and also in the mathematical literature, there are a large number of node-degree-based graph descriptors are being studied and used. The most commonly extensively investigated of them, are the first $M_1(\Gamma)$ and second $M_2(\Gamma)$ Zagreb indices. These two TIs are introduced and elaborated by Gutman and Trinajstic, since more than forty years [10, 9], and are gave by:

$$\begin{aligned} M_1(\Gamma) &= \sum_{x \in V(\Gamma)} d^2(x) = \sum_{yx \in E(\Gamma)} \Big(deg(x) + deg(y)\Big), \\ M_2(\Gamma) &= \sum_{yx \in E(\Gamma)} deg(y)deg(x). \end{aligned}$$

In 2008, Yamaguchi [20], depended on the second degree-sum of a graph to established new bounds on $M_1(\Gamma)$, $M_2(\Gamma)$ and the graph spectral radius $\rho(\Gamma)$, that is the maximum eigenvalue of a graph adjacency matrix $A(\Gamma)$.

In 2017, Akber and Trinajstic [1], defined a modified first Zagreb connection index depended on the second degrees of nodes. They formatted it as:

$$ZC_1^*(\Gamma) = \sum_{x\in V(\Gamma)} deg(x)deg_2(x).$$

Naji et al. (2017) [14], used second and fires degree of nodes to defined three new distance-degree-based TIs, and named them leap Zagreb indices. They defined as:

$$\begin{aligned} LM_1(\Gamma) &= \sum_{x\in V(\Gamma)} deg_2^2(x) \\ LM_2(\Gamma) &= \sum_{yx\in E(\Gamma)} deg_2(y)deg_2(x) \\ LM_3(\Gamma) &= \sum_{x\in V(\Gamma)} deg(x)deg_2(x). \end{aligned}$$

These invariants,are founded have many applications in chemistry. In particular, the first one has a vital role in the physical properties of chemical compounds, where it has approximation values to the entropy, boiling point,accentric factor and DHVAP, HVAP, for details see[3]. Thus these TIs attract attentions of the many graph researchers and also other scientists mostly chemists. For more details in the properties of these indices, see [2, 3, 11, 12, 13, 14, 15, 16, 17, 18, 19].

The notion of second degrees of a graph supplied us an additional area to study and analysis various structural properties of graphs. The second degree of nodes have been used in the definition of several parameters of graphs. For instance, in the 2-domination of a graph, which introduced by Fink and Jacobson [5, 6], and in others beyond 2-domination related concepts. Another important reason to study the second degree of a graph that is many graph theory researchers used them to define various topological indices of a graph.

In this paper, motivated by the notion of degree of nodes of a graph, we investigate properties of a second degree of a graph. Upper and lower bounds for second degree are presented. We provide an improvement on the useful result, that found in [19, 20], and state that "

$$deg_2(x) \leq \big(\sum_{y\in N(x)} deg(y)\big) - deg(x)$$

, for every $x \in V(\Gamma)$", by adding a term of the number of triangles in Γ, in order to the equality holds for each quadrangle-free graph. The general formulas of the second degrees of some graph operations will present Some lower and upper bounds on the maximum and minimum second degrees in graphs will be establish. Through, this work, we will derive the second-degree-sum formula as weel as we will establish upper and lower bounds on it .

2. Properties of second degrees

In this section, we interested the properties of the second degrees of a graph. In 2.1, we present the old bounds for second degree and we also establish some new bounds, in 2.2, the general formulas of second degrees of some graph operations are presented. In 2.3, the maximum and minimum second degrees of a graph and some bound on them are discussed. Finally, The properties of second degree sum is investigated.

2.1. Bounds for second degrees

For a graph Γ, the simple lower and upper bounds on the second degrees in a graph can be specified in terms of its order.

Lemma 2.1. *For every nodes x in a connected graph Γ,*

$$0 \leq deg_2(x) \leq p - 2. \tag{1}$$

The left inequality holds if and only if $deg(x) = p - 1$ whereas the right holds if and only if x is a pendent node whose support node has degree $p - 1$.

We are herewith represent the fundamental old bounds on the second degree which are collected from the papers [19, 14] and [20].

Lemma 2.2. *For every connected Γ, if $x \in V(\Gamma)$, then*

$$deg_2(x) \leq \Big(\sum_{y \in N(x)} deg(y) \Big) - deg(x). \tag{2}$$

Equality achieves if and only if x is not containing in C_3 or C_4 induced subgraph of Γ.

Lemma 2.3. *If Γ is a connected graph of order p, then for every node x in Γ*

$$deg_2(x) \leq p + 1 - deg(x) - e(x). \tag{3}$$

Equality is holding if and only if Γ is a Moore graph with $diam(\Gamma) \leq 2$.

Lemma 2.4. *Let Γ be a connected graph with n nodes. Then for every node $x \in V(\Gamma)$*

$$deg_2(x/\Gamma) \leq deg(x/\overline{\Gamma}) = p - 1 - deg(x/\Gamma). \tag{4}$$

with equality if and only if the eccentricity of x is at most two.

From Lemma 2.2, the following result follows,

Corollary 2.1. *If Γ is an r-regular (C_3, C_4)-free graph. Then*

$$deg_2(x) = r(r - 1). \tag{5}$$

In what follow, we provide an improvement on the second degree bound, Lemma 2.2, in order to the equality holds for every quadrangle-free graph. Let x be a node in Γ. Then $\tau(x)$ denote a triangles number containing in Γ, which have a common node x, and $\tau(\Gamma)$ denote the triangles total number of Γ.

Theorem 2.1. *Let Γ be a graph with p nodes and let $\tau(x)$ be the number of triangles in Γ joint with x. Then*

$$deg_2(x) \leq \Big(\sum_{y \in N(x)} deg(y) \Big) - deg(x) - 2\tau(x). \tag{6}$$

Equality holds, for every $x \in V(\Gamma)$, if and only if Γ is a quadrangle-free.

Proof. Let x be a node of a graph Γ, that is a common node of $\tau(x) \geq 0$ triangles in Γ. Suppose that H is the induced subgraph of Γ, which is constructed from a node x and all whose neighbors. That is $V(H) = \{x\} \cup N(x)$, and for a node $x \in V(\Gamma)$, let us denote by $deg(x/H)$ to the degree of x in H. Then for every $x \in V(\Gamma)$, $deg_i(x/\Gamma) = deg_i(x/H) + deg_i(x/\overline{H})$, for $i = 1, 2$.

According to the structure of a graph H, we have $deg(x) = deg(x/\Gamma) = deg(x/H)$, $deg_2(x) = deg_2(x/\Gamma) = deg_2(x/\overline{H})$, and so, for any a node $y \in N(x)$, $deg(y/\Gamma) = deg(y/H) + deg(y/\overline{H})$. Hence,

$$\sum_{y \in N(x)} deg(y/G) = \sum_{y \in N(x)} deg(y/H) + \sum_{y \in N(x)} deg(y/\overline{H}). \tag{7}$$

Now, for any two nodes $y_1, y_2 \in N(x)$, if $y_1y_2 \in E(H)$, then the induced subgraph by x, y_1 and y_2 is a triangle in Γ, and hence $|E(H)| = \tau(x) + deg(x)$. Since

$$2|E(H)| = \sum_{x \in V(H)} deg(x/H) = \big(\sum_{y \in N(x)} deg(y/H) \big) + deg(x).$$

Then

$$\sum_{y \in N(x)} deg(y/H) = 2\tau(x) + deg(x). \tag{8}$$

Correspondingly, since $N_2(x/\Gamma) = N_2(x/\overline{H}) = \bigcup_{y \in N(x)} N(y/\overline{H})$. Then with bring in mind Theorem of the cardinality of the sets union, we obtain that

$$\sum_{y \in N(x)} deg(y/\overline{H}) = deg_2(x) + \alpha, \tag{9}$$

where α is the sum of the cardinalities of the common neighbors for every two node y_1 and $y_2 \in N(x)$. Hence, by substitution equations (8) and (9) in (7), we obtain

$$\sum_{y \in N(x)} deg(y/\Gamma) = 2\tau(x) + deg(x) + deg_2(x) + \alpha \tag{10}$$

which implies, by omitting α, that

$$\sum_{y \in N(x)} deg(y/\Gamma) \geq 2\tau(x) + deg(x) + deg_2(x).$$

Therefore,

$$deg_2(x) \leq \Big(\sum_{y \in N(x)} deg(y/\Gamma) \Big) - deg(x) - 2\tau(x).$$

For equality, we have from equation (10), for any node $x \in V(\Gamma)$,

$$deg_2(x) = \Big(\sum_{y\in N(x)} deg(y/\Gamma)\Big) - deg(x) - 2\tau(x),$$

if and only if $\alpha = 0$, if and only if for any two vertice y_1 and $y_2 \in N(x)$, have no a common neighborhood in $\overline{H}$, if and only if a node x no containing in a quadrangle induced subgraph of Γ, if and only if Γ is a quadrangle-free graph. □

Proposition 2.1. *Let Γ be a quadrangle-free connected graph. Then*

$$\sum_{x\in V(\Gamma)} deg_2(x) = M_1(\Gamma) - 2q - 6\tau(\Gamma), \tag{11}$$

where $\tau(\Gamma)$ is the number of triangles in Γ.

Proof. From Theorem 2.1, we have if Γ is a quadrangle-free connected graph, then for each node $x \in V(\Gamma)$,

$$deg_2(x) = \Big(\sum_{y\in N(x)} deg(y)\Big) - deg(x) - 2\tau(x).$$

Tacking the sum on all nodes of Γ in both sides, and by using the facts that $\sum_{x\in V(\Gamma)} deg(x) = 2m$ and

$$\sum_{x\in V(\Gamma)} \Big(\sum_{y\in N(x)} deg(y)\Big) = \sum_{x\in V(\Gamma)} deg^2(x) = M_1(\Gamma).$$

Since the triangle has three nodes, each a node x of them share with $2\tau(x)$ in the formula. Then

$$\begin{aligned}
\sum_{x\in V(\Gamma)} deg_2(x) &= \sum_{x\in V(\Gamma)} \Big[\Big(\sum_{y\in N(x)} deg(y)\Big) - deg(x) - 2\tau(x)\Big] \\
&= \sum_{x\in V(\Gamma)} \Big(\sum_{y\in N(x)} deg(y)\Big) - \sum_{x\in V(\Gamma)} deg(x) - \sum_{x\in V(\Gamma)} 2\tau(x) \\
&= \sum_{x\in V(\Gamma)} deg^2(x) - 2q - 6\tau(\Gamma) \\
&= M_1(\Gamma) - 2q - 6\tau(\Gamma).
\end{aligned}$$

□

2.2. The second degree formulas of some graph operations

In this subsection, the general formula of the second degrees of some graph operations is collected from [13, 14]. For graphs Γ_1 and Γ_2 with disjoint node sets $V(\Gamma_1)$ and $V(\Gamma_2)$, and let us denote by $E(\Gamma_i)$, p_i and q_i to link set, order and size of Γ_i, for $i = 1, 2$.

1. The *union* $\Gamma_1 \cup \Gamma_2$ of Γ_1 and Γ_2, , is a graph with node set equals $V(\Gamma_1) \cup V(\Gamma_2)$ and link set equals $E(\Gamma_1) \cup E(\Gamma_2)$. Then

$$deg_2(x/\Gamma_1 \cup \Gamma_2) = \begin{cases} deg_2(x/\Gamma_1), & \text{if } x \in V(\Gamma_1); \\ deg_2(x/\Gamma_2), & \text{if } x \in V(\Gamma_2). \end{cases} \tag{12}$$

2. The *join* $\Gamma_1 + \Gamma_2$ of Γ_1 and Γ_2, , is a graph whose node set is $V(\Gamma_1) \cup V(\Gamma_2)$ and link set is $E(\Gamma_1) \cup E(\Gamma_2) \cup \{xy : y \in V(\Gamma_1)\ and\ x \in V(\Gamma_2)\}$. Then

$$deg_2(x/\Gamma_1 + \Gamma_2) = \begin{cases} (p_1 - 1) - deg_2(x/\Gamma_1), & \text{if } x \in V(\Gamma_1); \\ (p_2 - 1) - deg_2(x/\Gamma_2), & \text{if } x \in V(\Gamma_2). \end{cases} \tag{13}$$

3. The *corona* product $\Gamma_1 \circ \Gamma_2$ is a graph that obtained from Γ_1 and Γ_2 by taking p_1 copies of Γ_2 with one copy of Γ_1, and connecting the i^{th} node (or x_i) in Γ_1 to every node in the i^{th} copy of Γ_2. Hence,

$$deg_2(x/\Gamma_1 \circ \Gamma_2) = \begin{cases} deg_2(x/\Gamma_1) + p_2 deg(x/\Gamma_1), & \text{if } x \in V(\Gamma_1); \\ (p_2 - 1 - deg(x/\Gamma_2^i) - deg(x_i/\Gamma_1), & \text{if } x \in V(\Gamma_2^i). \end{cases} \tag{14}$$

4. The *cartesian* product $\Gamma_1 \Box \Gamma_2$, is a graph whose node set is $V\Gamma_1) \times V(\Gamma_2)$, and every two nodes $y = (y_1, y_2)$ and $x = (x_1, x_2)$ in $V(\Gamma_1) \times V(\Gamma_2)$ are adjacent, if and only if either ($y_1 = x_1$ and $y_2x_2 \in E(\Gamma_2)$) or ($y_2 = x_2$ and $y_1x_1 \in E(\Gamma)$). For every $y \in V(\Gamma_1)$ and $x \in V(\Gamma_2)$,

$$deg_2((x, y)/\Gamma_1 \Box \Gamma_2) = deg_2(y/\Gamma_1) + deg_2(x/\Gamma_2) + deg(y/\Gamma_1)deg(x/\Gamma_2). \tag{15}$$

5. The *direct* (or Kronecker) product of Γ_1 and Γ_2, denoted by $\Gamma_1 \otimes \Gamma_2$, is a graph on node set $V(\Gamma_1) \times V(\Gamma_2)$, and (y_1, x_1) adjacent with (y_2, x_2) if and only if $y_1y_2 \in E(\Gamma_1)$ and $x_1x_2 \in E(\Gamma_2)$. Thus

$$deg_2((x, y)/\Gamma_1 \otimes \Gamma_2) = deg_2(y/\Gamma_1) + deg_2(x/\Gamma_2) + deg_2(y/\Gamma_1)deg_2(x/\Gamma_2). \tag{16}$$

6. The *composition* (or lexicographic) product of Γ_1 and Γ_2 denoted $\Gamma_1[\Gamma_2]$ is a graph on node set $V(\Gamma_1) \times V(\Gamma_2)$ in which (y_1, x_1) is adjacent with (y_2, x_2) whenever (y_1 is adjacent with y_2) or ($y_1 = y_2$ and x_1 is adjacent with x_2).

$$deg_2((x, y)/\Gamma_1[\Gamma_2]) = p_2 deg_2(y/\Gamma_1) + deg_1(x/\Gamma_2). \tag{17}$$

7. The *disjunction* product of Γ_1 and Γ_2 denoted $\Gamma_1 \vee \Gamma_2$ is the graph with node set $V(\Gamma_1) \times V(\Gamma_2)$ in which (y_1, x_1) is adjacent with (y_2, x_2) whenever y_1 is adjacent with y_2 in Γ_1 or x_1 is adjacent with x_2 in Γ_2.

$$deg_2((x, y)/\Gamma_1 \vee \Gamma_2) = (p_1p_2 - 1) - p_2 deg_1(y/\Gamma_1) - p_1 deg_1(x/\Gamma_2) + deg_1(y/\Gamma_1)deg_1(x/\Gamma_2). \tag{18}$$

8. The *Symmetric difference* of Γ_1 and Γ_2 denoted $\Gamma_1 \oplus \Gamma_2$ is the graph with node set $V(\Gamma_1) \times V(\Gamma_2)$ in which (y_1, x_1) is adjacent with (y_2, x_2) whenever y_1 is adjacent with y_2 in Γ_1 or x_1 is adjacent with x_2 in Γ_2 but not both.

$$deg_2((x, y)/\Gamma_1 \oplus \Gamma_2) = (p_1p_2 - 1) - p_2 deg_1(y/\Gamma_1) - p_1 deg_1(x/\Gamma_2) + 2deg_1(y/\Gamma_1)deg_1(x/\Gamma_2). \tag{19}$$

2.3. The Maximum and minimum second degrees in a graph

The maximum and the minimum second degree among all second degrees of the nodes of a graph Γ, will be denote by $\Delta_2 = \Delta_2(\Gamma)$ and $\delta_2 = \delta_2(\Gamma)$, and are defined, respectively, by

$$\Delta_2 = \max\{deg_2(x) : x \in V(\Gamma)\} \text{ and } \delta_2 = \min\{deg_2(x) : x \in V(\Gamma)\}.$$

From Lemma 2.1, the following result straightforward,

Proposition 2.2. *Let Γ be a connected graph with n nodes. Then*

$$\Delta_2 \leq p - 2, \tag{20}$$

the bound attains if Γ has a pendent node whose support with $p-1$ degree.

From Lemma 2.4, the following result straightforward,

Proposition 2.3. *Let Γ be a connected graph with n nodes, and minimum degree δ. Then*

$$\Delta_2 \leq p - (\delta + 1), \tag{21}$$

the bound attains on a graph with $diam(\Gamma) \leq 2$.

From Lemma 2.3, the following result straightforward,

Proposition 2.4. *Let Γ be a connected graph with n nodes, minimum degree δ and radius $rad(\Gamma)$. Then*

$$\Delta_2 \leq n + 1 - \delta - rad(\Gamma), \tag{22}$$

the bound attains on C_4, C_5, P_4 and C_6.

Since, for any node $x \in V(\Gamma)$, $\delta_2 \leq deg_2(x) \leq \Delta_2$. Then every lower (upper) bound for δ_2 (Δ_2) is a lower (upper) bound for the second degree of a graph.

Theorem 2.2. *Let Γ be a connected graph with $n \geq 2$ nodes. Then*

$$\Delta_2 \leq 2\binom{\Delta}{2}. \tag{23}$$

Equality holds, if and only if Γ has a node x that is not containing in a triangle or quadrangle subgraph of Γ and $deg(x) = deg(y) = \Delta$, for every $y \in N(x)$.

Proof. For a node x of a graph Γ, we have from Lemma 2.2,

$$deg_2(x) \leq \Big(\sum_{y \in N(x)} deg(y)\Big) - deg(x).$$

Now, let x be the node of a graph Γ with the maximum second degree, i.e., $deg_2(x) = \Delta_2(\Gamma)$. Since for any node x in $V(\Gamma)$, $deg(x) \leq \Delta$ and $\sum_{y \in N(x)} deg(y) \leq \Delta deg(x)$. Then

$$\begin{aligned} \Delta_2 &\leq \Big(\sum_{y \in N(x)} deg(y) \Big) - deg(x) \\ &\leq \Big(\sum_{y \in N(x)} \Delta \Big) - deg(x) \\ &\leq \Delta deg(x) - deg(x) \\ &\leq deg(x)(\Delta - 1) \\ &\leq \Delta(\Delta - 1). \end{aligned}$$

Therefore $\Delta_2 \leq 2\Big(\frac{\Delta(\Delta-1)}{2}\Big) = 2\binom{\Delta}{2}$.

For equity, we have from Lemma 2.2, $deg_2(x) = \sum_{y \in N(x)} deg(y) - deg(x)$, if and only if a node x is not containing in a triangle or quadrangle induced subgraph of Γ. Thus $deg_2(x) = \Delta(\Delta - 1)$, if and only if $deg(x) = deg(y) = \Delta$, for every $y \in N(x)$.
Now, if x be a node of Γ with $deg_2(x) = \Delta_2$, then $\Delta_2 = \Delta(\Delta - 1)$, if and only if x is not containing in a triangle or quadrangle induced subgraph of Γ, and $deg(x) = deg(y) = \Delta$, for every $y \in N(x)$. $\square$

The following graph definition is required to illustrate the above Theorem.
Let Γ be a graph with p nodes and let $K_{1,p-1} * \Gamma$, be the graph obtained by taking one copy of the star $K_{1,p-1}$ and $p-1$ copies of Γ, and then joining the i^{th} pendant node (namely x_i) of the star to every node in the i^{th} copy (namely Γ_2^i) of Γ. For example the figure blow shown the graph $K_{1,4} * \overline{K_3}$.

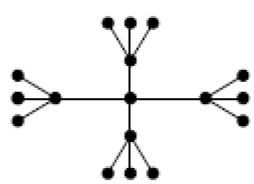

Figure 1. A graph $K_{1,4} * \overline{K_3}$

It is easy to see that, if Γ is a path P_p, for $p \neq 3, 4$, or a $K_{1,p-1} * H$, where H is a graph with $\Delta - 1$ nodes, or a C_3, C_4-free regular graph, then $\Delta_2(\Gamma) = \Delta(\Delta - 1)$.

Proposition 2.5. *Let Γ be a graph with $\delta \geq \frac{p-1}{2}$. Then $\Delta_2 \leq \frac{p-1}{2}$*

Proof. Since the minimum degree of Γ is $\delta \geq \frac{p-1}{2}$. Then the diameter of Γ is at most two. Thus by Proposition 2.2,

$$\begin{aligned}\Delta_2(\Gamma) &\leq p - 1 - \delta \\ &= p - 1 - \frac{p-1}{2} \\ &= \frac{p-1}{2}.\end{aligned}$$

□

2.4. The second degree-Sum Formula of a graph

Let $E_2(\Gamma)$ denote the set of all unorder pairs of nodes of Γ which the distance between them equal two, i.e., $E_2(\Gamma) = \Big\{\{y, x\} : d(x, y) = 2$, for every $x, y \in V(\Gamma)\Big\}$ and let $q_2 = |E_2(\Gamma)|$. Then the following our main result contains the second degree analogs of the degree sum formula, which says that in any graph the sum of the degrees of the nodes is equal to twice the number of links.

Theorem 2.3. *Let Γ be a connected graph with n nodes.Then*

$$\sum_{x \in V(\Gamma)} deg_2(x) = 2q_2. \tag{24}$$

Proof. Let Γ' be a graph constructed from Γ, such that $V(\Gamma') = V(\Gamma)$ and for every pair of nodes $x, y \in V(\Gamma)$, $xy \in E(\Gamma')$, if and only if $d(x, y) = 2$, in Γ, if and only if $xy \in E_2(\Gamma)$. Then $E_2(\Gamma) = E(\Gamma')$ and hence for every node $x \in V(\Gamma)$, $deg_2(x/\Gamma) = deg(x/\Gamma')$. Therefore

$$\sum_{x \in V(\Gamma)} deg_2(x/\Gamma) = \sum_{x \in V(\Gamma)} deg(x/\Gamma') = 2|E(\Gamma')| = 2|E_2(\Gamma)| = 2q_2.$$

□

From this Theorem, the following result is an analog of a familiar result in graph theory; namely, that every graph has an even number of odd nodes.

Corollary 2.2. *For a graph Γ, the number of nodes with odd second degree in Γ is even.*

Lemma 2.5. *Let Γ be a graph with p nodes and q_2 second links. Then*

1. *Every graph Γ has at least two nodes with the same second degree. That is there are at least $x, y \in V(\Gamma)$ such that $deg_2(x) = deg_2(y)$.*
2. *The average of second degrees in a graph Γ is $\frac{2q_2}{p}$ and*

$$\delta_2 \leq \frac{2q_2}{p} \leq \Delta_2. \tag{25}$$

Theorem 2.4. *Let Γ be a connected graph with p nodes, q links and q_2 second links. Then*

$$q_2 \leq \binom{p-1}{2}, \tag{26}$$

with equality if and only if Γ is a star.

Proof. From Lemma 2.1, we have $deg_2(x) \leq (p-2)$, for every node x in a graph (Γ) with p nodes. Since for a node x in Γ, $deg_2(x) = p-2$, if and only if $deg(x) = 1$ and $deg(y) = p-1$, for $\{y\} = N(x)$. Thus by taken the summation of the second degrees over all nodes of a graph Γ, and with bring in mind that Γ has at least a node x_0 with $deg(x_0) = p-1$, and $deg_2(x_0) = 0$, we obtain

$$\begin{aligned} 2q_2 = \sum_{x \in V(\Gamma)} deg_2(x) &\leq \sum_{x \in V(\Gamma)} (p-2) \\ &= deg_2(x_0) + \sum_{\substack{x \in V(\Gamma) \\ x \neq x_0}} (p-2) \\ &= 0 + (p-1)(p-2) \\ &= (p-1)(p-2). \end{aligned}$$

Therefore, $q_2 \leq \frac{(p-1)(p-2)}{2} = \binom{p-1}{2}$.

For equality, Since for a node x in a graph Γ, $deg_2(x) = p-2$, if and only if $deg(x) = 1$ and $deg(y) = p-1$, for $\{y\} = N(x)$. That is $2q_2 = (p-1)(p-2)$, if and only if Γ has $p-1$ nodes with second degree $p-2$, if and only if Γ has only one node x_0 with $deg(x_0) = p-1$ and each other node x in Γ with $deg(x) = 1$, if and only if Γ is a star. □

Remark 2.1. It is clear that, the result in Theorem 2.4, $q_2 \leq \binom{p-1}{2}$, is an analogue for a familiar result of the notation q (the number of links in a graph Γ with p nodes), namely that for any graph Γ with p nodes and q links, $q \leq \binom{p}{2}$, with equality if and only if Γ is complete.

From Lemma 2.4, the following result on the notation q_2 of a graph straightforward.

Proposition 2.6. *Let Γ be a connected graph with p nodes, q links and q_2 second links. Then*

$$q_2 \leq \frac{p(p-1)}{2} - q, \tag{27}$$

with equality, if and only if $diam(\Gamma) \leq 2$.

Corollary 2.3. *Let Γ be a connected graph with p nodes, q links and q_2 second links. Then*

$$q_2 \leq \frac{p(p-1-\delta)}{2}, \tag{28}$$

with equality, if and only if $diam(\Gamma) \leq 2$.

By Theorem 2.1, the following result is holding.

Proposition 2.7. *Let Γ be a connected graph with p nodes, q links and q_2 second links. Then*

$$q_2 \leq \frac{1}{2}M_1(\Gamma) - q - 3\tau(\Gamma), \tag{29}$$

where $\tau(\Gamma)$ is the number of triangles in Γ, and equality holds if and only if Γ is a quadrangle-free.

Corollary 2.4. *Let Γ be a graph with p nodes and q links. Then*

$$q_2 \leq \frac{1}{2}M_1(\Gamma) - q. \tag{30}$$

Equality holds if and only if Γ is C_3, C_4-free graph.

Proposition 2.8. *Let Γ be a connected graph with p nodes, q links and q_2 second links. Then*

$$q_2 \leq q(\Delta - 1), \tag{31}$$

with equality holds if and only if Γ is C_3, C_4-free Δ-regular graph.

Proof. From Lemma 2.2, we have $deg_2(x) \leq \Big(\sum\limits_{y\in N(x)} deg(y)\Big) - deg(x)$, for every node x in a graph Γ. Then we obtain that

$$\begin{aligned} 2q_2 = \sum_{x\in V(\Gamma)} deg_2(x) &\leq \sum_{x\in V(\Gamma)} \Big[\Big(\sum_{y\in N(x)} deg(y)\Big) - deg(x)\Big] \\ &\leq \sum_{x\in V(\Gamma)} \Big[\Big(\sum_{y\in N(x)} \Delta\Big) - deg(x)\Big] \\ &\leq \sum_{x\in V(\Gamma)} \Big[\Delta deg(x) - deg(x)\Big] \\ &\leq \sum_{x\in V(\Gamma)} \Big[deg(x)(\Delta - 1)\Big] \\ &= 2q(\Delta - 1). \end{aligned}$$

The equality is holding, if and only if $deg_2(x) = \Big(\sum\limits_{y\in N(x)} deg(y)\Big) - deg(x)$, and $deg(x) = \Delta$, for every node x in a graph Γ, if and only if Γ is C_3, C_4-free Δ-regular graph. □

Theorem 2.5. *For a connected graph Γ with p nodes and q links,*

$$q_2 \leq \frac{q(q-1)}{2}. \tag{32}$$

Equality holds if and only if Γ is a star.

Proof. From Proposition 2.6, we have $q_2 \leq \frac{p(p-1)}{2} - q$. Since for any connected graph Γ with p nodes and q links, $p - 1 \leq q$. Then

$$q_2 \leq \frac{q(q+1)}{2} - q = \frac{q(q+1) - 2q}{2} = \frac{q(q+1-2)}{2} = \frac{q(q-1)}{2}.$$

For equality, since the equality $q_2 = \frac{p(p-1)}{2} - m$, is holding, if and only if $diam(\Gamma) \leq 2$, and the equality $p - 1 = m$, is holding, if and only if Γ is a tree. Then the equality $q_2 = \frac{q(q-1)}{2}$, is holding if and only if, Γ is a tree with diameter at most two, if and only if Γ is a star. □

Theorem 2.6. *Let Γ be a graph with p nodes, q links and $\Delta(\Gamma) \neq p - 1$. Then*

$$q_2 \geq \frac{q\delta_2}{\Delta}. \tag{33}$$

Equality holds if and only if Γ is $(p - \delta_2 - 1)$-regular graph with $deg_2(x) = \delta_2$ for every $x \in V(\Gamma)$.

Proof. Let Γ be a graph with p nodes, q links and maximum degree $\Delta(\Gamma) \neq p - 1$. Then

$$\begin{aligned} 2q_2 = \sum_{x \in V(\Gamma)} deg_2(x) &= \sum_{yx \in E(\Gamma)} \left(\frac{deg_2(y)}{deg(y)} + \frac{deg_2(x)}{deg(x)} \right) \\ &\geq \sum_{yx \in E(\Gamma)} \left(\frac{\delta_2}{\Delta} + \frac{\delta_2}{\Delta} \right) \\ &= \sum_{yx \in E(\Gamma)} \frac{2\delta_2}{\Delta} = \frac{2q\delta_2}{\Delta}. \end{aligned}$$

Equality is holding if and only if $deg_2(x) = \delta_2$ and $deg(x) = \Delta = p - \delta_2 - 1$, for every $x \in V(\Gamma)$. This complete the proof. □

Corollary 2.5. *Let Γ be a graph with p nodes, q links and $\Delta(\Gamma) \neq p - 1$. Then*

$$q_2 \geq \frac{q}{\Delta}. \tag{34}$$

Equality holds if and only if Γ is $(p - 2)$-regular graph.

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