Indonesian Journal of Combinatorics **7** (1) (2023), 16–23

INDONESIAN JOURNAL OF COMBINATORICS

Γ-supermagic labeling of products of two cycles with cyclic groups

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Abstract

A Γ-supermagic labeling of a graph $G = (V, E)$ is a bijection from E to a group Γ of order $|E|$ such that the sum of labels of all edges incident with any vertex $x \in V$ is equal to the same element $\mu \in \Gamma$. A Z_{2mn}-supermagic labeling of the Cartesian product of two cycles, $C_m \Box C_n$ for every $m, n \geq 3$ was found by Froncek, McKeown, McKeown, and McKeown. In this paper we present a Z_k-supermagic labeling of the direct and strong product by cyclic group Z_k for any $m, n \geq 3$.

Keywords: Supermagic labeling, vertex-magic edge labeling, group supermagic labeling, product of cycles
Mathematics Subject Classification : 05C78

1. Introduction

A *supermagic labeling* of a graph $G = (V, E)$ is a bijection $f : E \to \{1, 2, \ldots, |E|\}$ with the property that at every vertex x the sum of the labels of all vertices incident with x is equal to the same constant c. When we replace the set of first $|E|$ positive integers by a group Γ of order $|E|$, we speak about a Γ-*supermagic labeling*.

The three most common products of graphs are the *Cartesian product* $G \Box H$, the *direct product* $G \times H$, and the *strong product* $G \boxtimes H$.

Supermagic labelings of Cartesian products of two cycles were studied by Ivančo in 2000 [6]. After almost a twenty year hiatus, the problem was revisited by various sets of authors who studied Γ-supermagic labelings of $C_m \Box C_n$ for a wide spectrum of Abelian groups Γ (see [2, 3, 4, 5, 7, 9]).

Received: 31 December 2022, Revised: 04 May 2023, Accepted: 17 May 2023.

The other two main graph product have not drawn any attention so far. We therefore initiate research in this direction by studying Z_k-supermagic labelings of the direct and strong products of two cycles. We present a construction for Z_{2mn}-supermagic labeling of $C_m \times C_n$ in Section 4 and for Z_{4mn}-supermagic labeling of $C_m \boxtimes C_n$ in Section 5.

Exact definitions of the above notions are given in Section 2. The results mentioned in the previous paragraph are listed in detain in Section 3.

Disclaimer. As noted above, the topic of this paper is very similar to the topic of [4] and [5]. Most of the known results cited in this paper have been also cited in these two papers and the statements of the cited theorems here are therefore identical. Also, some text in Sections 2 and 3 may be taken directly from [4] or [5].

2. Definitions

For the sake of completeness, we start with the definitions of various products of two graphs. We start with the *Cartesian product*.

Definition 2.1. The *Cartesian product* $G = G_1 \Box G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2, respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

Another well known product is the *direct product*, also called the *tensor* or *Kronecker product*.

Definition 2.2. The *direct product* $G = G_1 \times G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2, respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2.

The *strong product* is just the union of the above two.

Definition 2.3. The *strong product* $G = G_1 \boxtimes G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2, respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if u is adjacent to v in $G_1 \Box G_2$ or u is adjacent to v in $G_1 \times G_2$.

The notion of *supermagic labeling* was also studied under the name of *vertex-magic edge labeling.*

Definition 2.4. A *supermagic labeling* of a graph $G(V, E)$ with $|E| = q$ is a bijection f from E to the set $\{1, 2, \ldots, q\}$ such that the sum of labels of all incident edges of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same positive constant c, called the *magic constant*. That is,

$$w(x) = \sum_{xy \in E} f(xy) = c$$

for every vertex $x \in V$. A graph that admits a supermagic labeling is called a *supermagic graph*.

There were also some more general forms of edge labelings studied by Sedláček [8] and Stanley [10, 11]. Stewart [12] introduced the notion of supermagic labeling, where the set of labels consisted of $|E|$ consecutive integers. When a supermagic graph is regular, then the edge labels can start with any positive integer, and therefore are always considered to be $1, 2, \dots, |E|$.

Moving from the set of consecutive integers to groups of order $|E|$, we define a Γ-*supermagic labeling*.

Definition 2.5. A Γ-*supermagic labeling* of a graph $G(V, E)$ with $|E| = q$ is a bijection f from E to a group Γ of order q such that for every vertex $x \in V$ and its incident edges $e_1, e_2, \dots, e_r$ there exists an ordering $e_{i_1}, e_{i_2}, \dots, e_{i_r}$ for which the *weight* of x, denoted $w(x)$ and defined as

$$w(x) = f(e_{i_r})f(e_{i_{r-1}})\dots f(e_{i_1})$$

is equal to the same element $\mu \in \Gamma$, called the *magic constant*. A graph that admits a Γ-supermagic labeling is called a Γ-*supermagic graph*.

In this note, we only deal with Abelian groups. Because Abelian groups are commutative, we do not have to worry about the order of the edges in the weight function. Therefore, we simplify the above definition for Abelian groups as follows.

Definition 2.6. Let Γ be a finite additive Abelian group or order k. A Γ-*supermagic labeling* of a graph $G(V, E)$ with $|E| = k$ is a bijection $f : E \to \Gamma$ such that for every vertex $x \in V$ the *weight* of x defined as

$$w(x) = \sum_{xy \in E} f(xy)$$

is equal to the same element $\mu \in \Gamma$, called the *magic constant*. A graph that admits a Γ-supermagic labeling is called a Γ-*supermagic graph*.

3. Known results

Research in this area was initiated by Ivančo [6] who investigated labelings with positive integers. He proved two results.

Theorem 3.1 ([6]). *Let $n \geq 3$. Then the Cartesian product $C_n \square C_n$ has a supermagic labeling.*

Theorem 3.2 ([6]). *Let $m, n \geq 4$ be even integers. Then $C_m \square C_n$ has a supermagic labeling.*

Ivančo also conjectured that a supermagic labeling exists for all Cartesian products $C_m \square C_n$.

Conjecture 1 ([6]). *The Cartesian product $C_m \square C_n$ allows a supermagic labeling for any $m, n \geq 3$.*

Froncek in an unpublished manuscript [1] verified that the conjecture is true also when m, n are both odd and not relatively prime.

Theorem 3.3 ([1]). *Let $m, n \geq 3$ be odd integers and $\gcd(m, n) > 1$. Then $C_m \square C_n$ has a supermagic labeling.*

Froncek, McKeown, McKeown, and McKeown [2] proved a result analogical to Theorems 3.2 and 3.3 for the cyclic group Z_{2mn} where at least one of m, n is odd.

Theorem 3.4 ([2]). *The Cartesian product $C_m \Box C_n$ admits a Z_{2mn}-supermagic labeling for all odd $m \geq 3$ and any $n \geq 3$.*

Notice that Theorem 3.2 implies the existence of Z_{2mn}-supermagic labeling for m, n both even. Therefore, a complete characterization was established.

Later, Froncek and McKeown [3] used a different construction to prove the complete result and showed that the labeling is different from the one obtained in the proof of the previous theorem.

Theorem 3.5 ([2]). *The Cartesian product $C_m \Box C_n$ admits a Z_{2mn}-supermagic labeling for all $m, n \geq 3$.*

The construction from [3] was then used by Sorensen [9] and Paananen [7] to obtain a slightly more general result.[1] Notice that when mn is even, the group used in the theorem is not cyclic.

Theorem 3.6 ([7, 9]). *For any $m, n \geq 3$, the Cartesian product $C_m \Box C_n$ admits a Γ-supermagic labeling for $\Gamma = Z_{mn} \oplus Z_2$.*

Paananen [7] and Sorensen [9] also proved some more partial results that were later generalized by Froncek, Paananen, and Sorensen [4, 5].

Theorem 3.7 ([4, 5]). *Let $m, n \geq 3$ and $m \equiv n \pmod 2$. Then the Cartesian product $C_m \Box C_n$ admits a Γ-supermagic labeling by any Abelian group Γ of order $2mn$.*

The case of $m \equiv n + 1 \pmod 2$ remains open except for the groups Z_{2mn} and $Z_{mn} \oplus Z_2$.

4. Direct products

It is well known (see [13]) that when at least one of m, n is odd, then $C_m \times C_n$ is connected, and when m, n are both even, then it contains two components.

We denote the vertices by $x_{i,j}$ where $0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$. We also define *forward edges* $d_{i,j} = x_{i,j}x_{i+1,j+1}$ and *backward edges* $b_{i,j} = x_{i,j}x_{i+1,j-1}$ for all admissible i, j.

A vertex $x_{i,j}$ is then incident with forward edges $d_{i,j} = x_{i,j}x_{i+1,j+1}$ and $d_{i-1,j-1} = x_{i-1,j-1}x_{i,j}$, backward edges $b_{i,j} = x_{i,j}x_{i+1,j-1}$ and $b_{i-1,j+1} = x_{i-1,j+1}x_{i,j}$ and has neighbors $x_{i-1,j-1}, x_{i+1,j+1}, x_{i+1,j-1}, x_{i-1,j+1}$.

Construction 4.1. *Z_{2mn}-supermagic labeling of $C_m \times C_n$.*

Each column contains m forward edges and m backward edges.

We label the forward edges in each column consecutively with elements of the coset $\langle 2n \rangle + j$ in increasing order, and the backward edges consecutively with elements of the coset $\langle 2n \rangle - (j + 1)$ in decreasing order.

[1] Paananen [7] (2021) and Sorensen [9] (2020) worked on a joint project for their MS theses. While all results cited here are their joint work, their theses were written and defended independently. Both theses contain Theorem 3.6.

In particular, in column j we have

$$f(d_{i,j}) = j + 2ni$$

and

$$f(b_{i,j}) = -j - 2ni.$$

Because vertex $x_{i,j}$ is incident with forward edges $d_{i,j}$ and $d_{i-1,j-1}$ and backward edges $b_{i,j}$ and $b_{i-1,j+1}$,

$$\begin{aligned} w(x_{i,j}) &= f(d_{i-1,j-1}) + f(d_{i,j}) + f(b_{i,j}) + f(b_{i-1,j+1}) \\ &= (j - 1 + 2n(i-1)) + (j + 2ni) + (-j - 2ni) + (-j - 1 - 2n(i-1)) \\ &= -2. \end{aligned}$$

Clearly, the weight is constant for each vertex and the labeling is Z_{2mn}-supermagic.

The theorem below follows immediately for the above construction.

Theorem 4.2. *Let $m, n \geq 3$ and Z_{2mn} be the cyclic group of order $2mn$. Then there exists a Z_{2mn}-supermagic labeling of the direct product $C_m \times C_n$.*

5. Strong products

We use the same vertex notation as in Section 4. Because the strong product $C_m \boxtimes C_n$ is in fact the union of the direct product $C_m \times C_n$ and the Cartesian product $C_m \Box C_n$, for the forward and backward edges we will use the labeling found in Construction 4.1.

For the vertical and horizontal edges arising from the Cartesian product, we first introduce diagonals. We call the cycle induced by vertices $x_{0,0}x_{0,1}, x_{1,1}, x_{1,2}, \ldots, x_{m-1,n-1}, x_{m-1,0}, x_{0,0}$ the zero diagonal and denote it by D^0. It should be obvious that the length of D^0 is $2\mathrm{lcm}(m,n)$. We denote $l = \mathrm{lcm}(m,n)$.

Therefore, there are in total $g = \gcd(m,n)$ such diagonals. We will call the diagonal induced by vertices $x_{0,j}x_{0,j+1}, x_{1,j+1}, x_{1,j+2}, \ldots, x_{m-1,j-1}, x_{m-1,j}, x_{0,j}$ for $1 \leq j \leq g-1$ the *j-th diagonal* and denote it by D^j.

To simplify notation, we denote the horizontal and vertical edges of D^j by h_k^j and v_k^j, respectively, where $0 \leq k \leq l-1$. The diagonal then has edges $h_0^j, v_0^j, h_1^j, \ldots, h_{l-1}^j, v_{l-1}^j$.

Construction 5.1. *Z_{2mn}-supermagic labeling of $C_m \boxtimes C_n$.*

We denote our labeling function by g and for the forward and backward edges we use the labeling function f from Construction 4.1 and define

$$g(d_{i,j}) = 2f(d_{i,j})$$

and

$$g(b_{i,j}) = 2f(b_{i,j}).$$

We define the *partial DB-weight* $w_{DB}(x_{i,j})$ of a vertex $x_{i,j}$ as the sum of labels of the forward and backward edges incident with $x_{i,j}$, that is,

$$w_{DB}(x_{i,j}) = g(d_{i-1,j-1}) + g(d_{i,j}) + g(b_{i,j}) + g(b_{i-1,j+1}) = -4.$$

So far, we have used all even labels. For the remaining edges we use odd labels. We label the horizontal edges of D^j consecutively by the elements of the coset $\langle 2l \rangle + 2j + 1$ in increasing order and the vertical edges by the elements of the coset $\langle 2l \rangle - 2j - 1$ in decreasing order. More precisely, we have

$$g(h_i^j) = 2j + 1 + 2li$$

and

$$g(v_i^j) = -2j - 1 - 2li.$$

For each diagonal, we define two partial weights as follows:

$$w_{HV}(x_{s,t}) = g(h_i^j) + g(v_i^j)$$

and

$$w_{VH}(x_{s,t}) = g(v_{i-1}^{j+1}) + g(h_i^{j+1})$$

where the edges in the labeling functions are incident with the vertex $x_{s,t}$.

Hence, for any permissible s and t we have

$$\begin{aligned} w_{HV}(x_{s,t}) &= g(h_i^j) + g(v_i^j) \\ &= (2j + 1 + 2li) + (-2j - 1 - 2li) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} w_{VH}(x_{s,t}) &= g(v_{i-1}^{j+1}) + g(h_i^{j+1}) \\ &= (-2j - 3 - 2l(i-1)) + (2j + 3 + 2li) \\ &= 2, \end{aligned}$$

regardless of the location of vertex $x_{s,t}$.

The total weight of each vertex $x_{s,t}$ is then

$$\begin{aligned} w(x_{s,t}) &= w_{DB}(x_{s,t}) + w_{HV}(x_{s,t}) + w_{VH}(x_{s,t}) \\ &= -4 + 0 + 2 \\ &= -2 \end{aligned}$$

and the labeling is Z_{4mn}-distance magic.

By constructing the labeling above, we proved the following.

Theorem 5.2. *Let $m, n \geq 3$ and Z_{4mn} be the cyclic group of order $4mn$. Then there exists a Z_{4mn}-supermagic labeling of the strong product $C_m \boxtimes C_n$.*

6. Conclusion

Based on Theorems 4.2 and 5.2 and results on Cartesian products from [2] and [6], the following result holds.

Theorem 6.1. *There exists a Z_{2mn}-supermagic labeling of the Cartesian product $C_m \square C_n$ and the direct product $C_m \times C_n$, and a Z_{4mn}-supermagic labeling of the strong product $C_m \boxtimes C_n$ for every $m, n \geq 3$.*

It would be a natural next step to study Γ-supermagic labelings of $C_m \times C_n$ and $C_m \boxtimes C_n$ for other Abelian groups Γ. Another direction is to look at products of more than two cycles.

References

[1] D. Froncek, Supermagic labelings of $C_n \square C_m$, unpublished manuscript.

[2] D. Froncek, J. McKeown, J. McKeown, M. McKeown, Z_{2nm}-supermagic labeling of $C_n \square C_m$, *Indones. J. Combin.* **2(2)** (2018), 57–71.

[3] D. Froncek, M. McKeown, Note on diagonal construction of Z_{2nm}-supermagic labeling of $C_n \square C_m$, *AKCE Int. J. Graphs Comb.* **17(3)** (2020), 952–954.

[4] D. Froncek, P. Paananen, L. Sorensen, Group-supermagic labeling of Cartesian product of two odd cycles, *Bull. Inst. Combin. Appl.*, accepted.

[5] D. Froncek, P. Paananen, L. Sorensen, Group-supermagic labeling of Cartesian product of two even cycles, submitted.

[6] J. Ivančo, On supermagic regular graphs, *Math. Bohem.* **125** (2000), 99–114.

[7] P. Paananen, Γ-*supermagic Labeling of* $C_m \square C_n$, MS Thesis, University of Minnesota Duluth, Duluth, MN, U.S.A., 2021.

[8] J. Sedláček, Problem 27, in: M. Fiedler (Ed.), Theory of Graphs and Its Applications, Praha, 1964, pp.163–164.

[9] L. Sorensen, Γ-*supermagic Labeling of* $C_m \square C_n$, MS Thesis, University of Minnesota Duluth, Duluth, MN, U.S.A., 2020.

[10] R. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.* **40** (1973), 607–632.

[11] R. Stanley, Magic labelings of graphs, symmetric magic squares, systems of parameters, and Cohen-Macaulay rings, *Duke Math. J.* **43** (1976), 511–531.

[12] B. M. Stewart, Magic graphs, *Canad. J. Math.* **18** (1996), 1031–1059.

[13] P. M. Weichsel, The Kronecker product of graphs, *Proc. Amer. Math. Soc.* **13** (1962), 47–52.