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A note on vertex irregular total labeling of trees

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Abstract

The total vertex irregularity strength of a graph $G = (V, E)$ is the minimum integer k so that there is a mapping from $V \cup E$ to the set $\{1, 2, \ldots, k\}$ for which the vertex-weights (i.e., the sum of labels of a vertex together with the edges incident to it) are all distinct. In this note, we present a new sufficient condition for a tree to have total vertex irregularity strength $\lceil (n_1 + 1)/2 \rceil$, where n_1 is the number of vertices of degree one in the tree.

Keywords: vertex irregular total k-labeling, total vertex irregularity strength, trees
Mathematics Subject Classification : 05C78

1. Introduction

Here, all graphs considered are only finite and undirected containing no loops nor multiple edges. Let G be a graph with vertex set V and edge set E. The degree of a vertex x is denoted by $\deg(x)$. The maximum and minimum degree of vertices of G are denoted by Δ and δ, respectively.

In 2007, Bača et al. [1] introduced a vertex irregular total labeling of a graph as an extension of an irregular labeling defined by Chartrand et al. [2]. For a positive integer k, a total k-labeling $\varphi : V \cup E \to \{1, 2, \ldots, k\}$ of a graph G is said to be a *vertex irregular total k-labeling* of G if $wt(x) \neq wt(y)$ for any two distinct vertices x, y where the *weight* of a vertex x is defined

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by $wt(x) = \varphi(x) + \sum_{xz \in E} \varphi(xz)$. The least integer k so that G admits a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G and denoted by $\mathrm{tvs}(G)$.

In [4], Nurdin et al. gave a general lower bound for the total vertex irregularity strength of an arbitrary tree T with maximum degree Δ:

$$\mathrm{tvs}(T) \geqslant \max\{t_i : i = 1, 2, \ldots, \Delta\}, \tag{1}$$

where $t_i = \lceil (1 + \sum_{j=1}^{i} n_j)/(i+1) \rceil$, and n_j denotes the number of vertices of degree j. In the same paper, they proposed a conjecture stating that the total vertex irregularity strength of any tree is determined only by the number of vertices of degree one, two, and three in the tree.

Conjecture 1. *[4] For every tree T with maximum degree Δ,* $\mathrm{tvs}(T) = \max\{t_1, t_2, t_3\}$.

This conjecture has been verified to be true for trees without vertices of degree two [4], irregular subdivision of trees [6], and trees with maximum degree four and five [7, 8]. In [5], Simanjuntak, Susilawati and Baskoro studied the total vertex irregularity strength of trees with many vertices of degree two and provided some sufficient conditions for trees to have total vertex irregularity strength t_1, t_2 or t_3. Specifically, they proved the following theorem.

Theorem 1.1. *[5] Let T be a tree. If $n_2 \leqslant \frac{n_1+1}{2}$ and $n_2 = n_3 > 0$ then* $\mathrm{tvs}(T) = t_1$.

In this note, we present another sufficient condition for a tree T to have $\mathrm{tvs}(T) = t_1$. In this new condition, we do not require n_2 and n_3 in T to be equal. In addition, we apply a slightly different algorithm to construct a vertex irregular total t_1-labeling of T.

The following property, found in [3], plays an important role in determining the total vertex irregularity strength of a tree, that is, for every tree T with maximum degree Δ,

$$n_1 = 2 + \sum_{i=3}^{\Delta} (i-2) n_i. \tag{2}$$

Consequently, for $i = 4, 5, \ldots, \Delta$,

$$n_i = \frac{n_1 - n_3 - 2 - \sum_{j=4, j \neq i}^{\Delta} (j-2) n_j}{i-2} < t_1. \tag{3}$$

2. Main results

Let us begin with the following lemma which reduces the number of variables appeared in (1).

Lemma 2.1. *For every tree T with maximum degree Δ,* $\max\{t_i : i = 1, 2, \ldots, \Delta\} = \max\{t_1, t_2, t_3\}$.

Proof. Consider $t_i - t_j$ for $1 \leqslant i < j \leqslant \Delta$ as follows.

$$\begin{aligned} t_i - t_j &= \left\lceil \frac{1 + \sum_{k=1}^{i} n_k}{i+1} \right\rceil - \left\lceil \frac{1 + \sum_{k=1}^{j} n_k}{j+1} \right\rceil \\ &= \left\lceil \frac{1 + n_1 + n_2 + \sum_{k=3}^{i} n_k}{i+1} \right\rceil - \left\lceil \frac{1 + n_1 + n_2 + \sum_{k=3}^{j} n_k}{j+1} \right\rceil \\ &= \left\lceil \frac{(j+1)(1 + n_1 + n_2 + \sum_{k=3}^{i} n_k)}{(i+1)(j+1)} \right\rceil - \left\lceil \frac{(i+1)(1 + n_1 + n_2 + \sum_{k=3}^{j} n_k)}{(i+1)(j+1)} \right\rceil . \end{aligned}$$

By substituting (2) to the above equation we get

$$\begin{aligned}
t_i - t_j = &\left\lceil \frac{(j+1)(3+n_2+\sum_{k=3}^{\Delta}(k-2)n_k+\sum_{k=3}^{i} n_k)}{(i+1)(j+1)} \right\rceil \\
&- \left\lceil \frac{(i+1)(3+n_2+\sum_{k=3}^{\Delta}(k-2)n_k+\sum_{k=3}^{j} n_k)}{(i+1)(j+1)} \right\rceil \\
= &\left\lceil \frac{(j+1)(3+n_2+\sum_{k=3}^{i}(k-1)n_k+\sum_{k=i+1}^{j}(k-2)n_k+\sum_{k=j+1}^{\Delta}(k-2)n_k)}{(i+1)(j+1)} \right\rceil \\
&- \left\lceil \frac{(i+1)(3+n_2+\sum_{k=3}^{i}(k-1)n_k+\sum_{k=i+1}^{j}(k-1)n_k+\sum_{k=j+1}^{\Delta}(k-2)n_k)}{(i+1)(j+1)} \right\rceil .
\end{aligned}$$

By putting $q_1 = 3+n_2+\sum_{k=3}^{i}(k-1)n_k+\sum_{k=j+1}^{\Delta}(k-2)n_k$ and $q_2 = \sum_{k=i+1}^{j}(k-1)n_k$, the above expression can be written as

$$t_i - t_j = \left\lceil \frac{(j+1)\left(q_1+q_2-\sum_{k=i+1}^{j} n_k\right)}{(i+1)(j+1)} \right\rceil - \left\lceil \frac{(i+1)(q_1+q_2)}{(i+1)(j+1)} \right\rceil . \tag{4}$$

Next we shall show that there is some i, $i \in \{1,2,3\}$, so that $t_i \geqslant t_j$ for $1 \leqslant j \leqslant \Delta$. The case $1 \leqslant j \leqslant 3$ is obvious. Suppose $j = 4$. If $t_2 \geqslant t_4$ then we are done. Assume now $t_2 < t_4$. We will show that $t_3 \geqslant t_4$. From (4) we obtain

$$t_2 - t_4 = \left\lceil \frac{5\,(q_1+q_2-n_3-n_4)}{15} \right\rceil - \left\lceil \frac{3(q_1+q_2)}{15} \right\rceil < 0,$$

so

$$5(q_1+q_2-n_3-n_4) - 3(q_1+q_2) < 0 \quad \Leftrightarrow \quad n_3 > 6+2n_2+n_4+2\sum_{k=5}^{\Delta}(k-2)n_k.$$

This implies that

$$\begin{aligned}
5(q_1+q_2-n_4)-4(q_1+q_2) =& 3+n_2+2n_3+\sum_{k=5}^{\Delta}(k-2)n_k+3n_4-5n_4 \\
>& 3+n_2+2\left(6+2n_2+n_4+2\sum_{k=5}^{\Delta}(k-2)n_k\right) \\
&+\sum_{k=5}^{\Delta}(k-2)n_k-2n_4 = 15+5n_2+5\sum_{k=5}^{\Delta}(k-2)n_k > 0.
\end{aligned}$$

Combining with (4), we get $t_3 \geqslant t_4$.

For the case $5 \leqslant j \leqslant \Delta$ one gets

$$(j+1)\left(q_1+q_2-\sum_{k=4}^{j} n_k\right)-4(q_1+q_2) = (j-3)q_1+(j-3)\sum_{k=4}^{j}(k-1)n_k-(j+1)\sum_{k=4}^{j} n_k$$
$$= (j-3)q_1+\sum_{k=4}^{j}((j-3)(k-2)-4)n_k > 0.$$

Combining with (4), we have $t_3 \geqslant t_j$. □

Lemma 2.2. *For every tree T of order at least three with $3n_3-n_1-1 \leqslant n_2 \leqslant \frac{n_1+1}{2}$ or $n_2 \leqslant 3n_3-n_1-2 \leqslant n_1-n_3+1$, we have that $t_1 = \max\{t_1,t_2,t_3\}$.*

Proof. First, suppose $3n_3-n_1-1 \leqslant n_2 \leqslant \frac{n_1+1}{2}$. As $n_2 \leqslant \frac{n_1+1}{2}$ we get

$$t_2 = \left\lceil \frac{n_1+n_2+1}{3} \right\rceil = \left\lceil \frac{2n_1+2n_2+2}{6} \right\rceil \leqslant \left\lceil \frac{2n_1+2(\frac{n_1+1}{2})+2}{6} \right\rceil = t_1.$$

Furthermore, since $n_2 \geqslant 3n_3-n_1-1$ we have $3n_3 \leqslant n_1+n_2+1$. So

$$t_3 = \left\lceil \frac{n_1+n_2+n_3+1}{4} \right\rceil \leqslant \left\lceil \frac{3n_1+3n_2+n_1+n_2+1+3}{12} \right\rceil = t_2 \leqslant t_1.$$

Therefore $t_1 = \max\{t_1,t_2,t_3\}$.

Now let $n_2 \leqslant 3n_3-n_1-2 \leqslant n_1-n_3+1$. As $n_2 \leqslant n_1-n_3+1$ we obtain $n_3 \leqslant n_1-n_2+1$. Therefore

$$t_3 = \left\lceil \frac{n_1+n_2+n_3+1}{4} \right\rceil \leqslant \left\lceil \frac{n_1+n_2+n_1-n_2+1+1}{4} \right\rceil = t_1.$$

Next, since $n_2 \leqslant 3n_3-n_1-2$ we get

$$t_2 = \left\lceil \frac{n_1+n_2+1}{3} \right\rceil \leqslant \left\lceil \frac{4n_1+3n_2+3n_3-n_1-2+4}{12} \right\rceil \leqslant \left\lceil \frac{3n_1+3n_2+3n_3+3}{12} \right\rceil = t_3 \leqslant t_1.$$

Thus $t_1 = \max\{t_1,t_2,t_3\}$. □

Lemma 2.3. *For every tree T of maximum degree $\Delta \geqslant 2$ with $3n_3-n_1-1 \leqslant n_2 \leqslant \frac{n_1+1}{2}$ or $n_2 \leqslant 3n_3-n_1-2 \leqslant n_1-n_3+1$, we have that $n_i \leqslant t_1$ for $i = 2,3,\ldots,\Delta$.*

Proof. According to (3), it remains to show that $n_i \leqslant t_1$ for $i = 2,3$. Let us first consider $3n_3-n_1-1 \leqslant n_2 \leqslant \frac{n_1+1}{2}$. Then $n_2 \leqslant \frac{n_1+1}{2} \leqslant t_1$, and since $3n_3-n_1-1 \leqslant \frac{n_1+1}{2}$ we have $n_3 \leqslant \frac{n_1+1}{2} \leqslant t_1$.

Now let $n_2 \leqslant 3n_3-n_1-2 \leqslant n_1-n_3+1$. As $3n_3-n_1-2 \leqslant n_1-n_3+1$ we get $n_3 \leqslant \frac{n_1+1}{2}+\frac{1}{4}$. However, n_3 is an integer and so $n_3 \leqslant \frac{n_1+1}{2} \leqslant t_1$. Furthermore, $n_2 \leqslant 3n_3-n_1-2 \leqslant 3(\frac{n_1+1}{2})-n_1-2 = \frac{n_1-1}{2} < t_1$. □

Let T be a tree. A vertex in T is called a *pendant vertex* if it has degree one. A *pendant edge* is an edge incident to a pendant vertex. An *exterior vertex* is a vertex adjacent to a pendant vertex. Every edge which is not pendant edge is called an *interior edge*. In the following theorem, we give a sufficient condition for a tree T with large number of exterior vertices to have $tvs(T) = t_1$.

Theorem 2.1. *Suppose T be a tree of order at least three with $3n_3 - n_1 - 1 \leqslant n_2 \leqslant \frac{n_1+1}{2}$ or $n_2 \leqslant 3n_3 - n_1 - 2 \leqslant n_1 - n_3 + 1$, and $n_2^e \geqslant 0$. If T contains n_2^e exterior vertices of degree two and contains at least $t_1 - 2n_2^e - 1$ exterior vertices of degree at least three then $tvs(T) = t_1$.*

Proof. It follows from (1), and Lemmas 2.1 and 2.2 that $tvs(T) \geqslant t_1$. To prove the equality, we provide a vertex irregular total t_1-labeling of T. Let us define a total labeling φ on vertices and edges of T using the following steps.

1. Let $V_{Ex} = \{v_1, v_2, \ldots, v_s\}$ be the set of s exterior vertices of T so that for every $i < j$, the following properties hold:
   (a) $\deg(v_i) \leqslant \deg(v_j)$.
   (b) If $\deg(v_i) = \deg(v_j)$ then $|E(v_i)| \geqslant |E(v_j)|$, where $E(v_i)$ denotes the set of pendant vertices adjacent to v_i.
2. For $j = 1, 2, \ldots, |E_P(v_i)|$ denote by v_{ij} the jth pendant vertex adjacent to the exterior vertex v_i. Denote by e_{ij} a pendant edge incident to v_{ij}. We then set $t := \max\{|E_P(v_i)| : i = 1, 2, \ldots, s\}$. For $j = 1, 2, \ldots, t$ let $V_P^j = \{v_{ij} : i = 1, 2, \ldots, s \text{ and } |E_P(v_i)| \geqslant j\}$. Denote by V_P the ordered set of union $\bigcup_{j=1}^{t} V_P^j$ where the order follows the original order in each V_P^j. Let also denote E_P as an ordered set of pendant edges so that e_{ij} is the kth element in E_P if and only if v_{ij} is the kth element in V_P.
3. Assign by 1 the first t_1 pendant vertices in V_P and by $2, 3, \ldots, n_1 - t_1 + 1$, respectively, the remaining pendant vertices in V_P. Then, assign by $1, 2, \ldots, t_1$, respectively, the first t_1 pendant edges in E_P and by t_1 the remaining pendant edges in E_P.
4. Assign by t_1 all interior edges of T.
5. Denote by $x_1, x_2, \ldots, x_N$, $N = |V| - n_1$, all the non-pendant vertices of T so that $\omega(x_i) \leqslant \omega(x_{i+1})$ for each i, where $\omega(x) := \sum_{xy \in E} \varphi(xy)$ denotes the temporary weight of x. We then define recursively:

$$\varphi(x_1) = \max\{1, n_1 + 2 - \omega(x_1)\}, \quad wt(x_1) = \varphi(x_1) + \omega(x_1),$$
$$\varphi(x_i) = \max\{1, wt(x_{i-1}) + 1 - \omega(x_i)\} \quad \text{for } i = 2, 3, \ldots, N.$$

We shall show that φ is a vertex irregular total t_1-labeling of T. It follows from the construction above that the weights of pendant vertices constitute the consecutive integers from 2 up to $n_1 + 1$, and for the weights of non-pendant vertices we have $n_1 + 2 \leqslant wt(x_1) < wt(x_2) < \cdots < wt(x_N)$. So all vertices of T have distinct weights.

It remains to prove that the largest label being used is t_1. It is easy to see from steps 3 dan 4 that all the pendant vertices and all the edges of T get labels at most t_1. Now, we show that every non-pendant vertex receive labels at most t_1, that is $\varphi(x_i) \leqslant t_1$ for $i = 1, 2, \ldots, N$.

Since T contains at least $t_1 - n_2^e - 1$ exterior vertices, one can verify that every vertex of degree $\partial \geqslant 2$ has temporary weight at least $(\partial - 1)t_1 + 1$, and no two distinct vertices with distinct degrees have identical temporary weights. Furthermore, if two vertices x and y have identical temporary weights then $\deg(x) = \deg(y) = \partial$ and $\omega(x) = \omega(y) = t_1\partial$, and by Lemma 2.3, $n_i \leqslant t_1$ for $i = 2, 3, \ldots, \Delta$, so there are at most t_1 such vertices. Therefore, the maximum label contributing to the corresponding final weights must be at most t_1. Hence φ is a vertex irregular total t_1-labeling of T, and we are done. $\square$

An example of vertex irregular total labeling of a tree is illustrated in Figure 1.

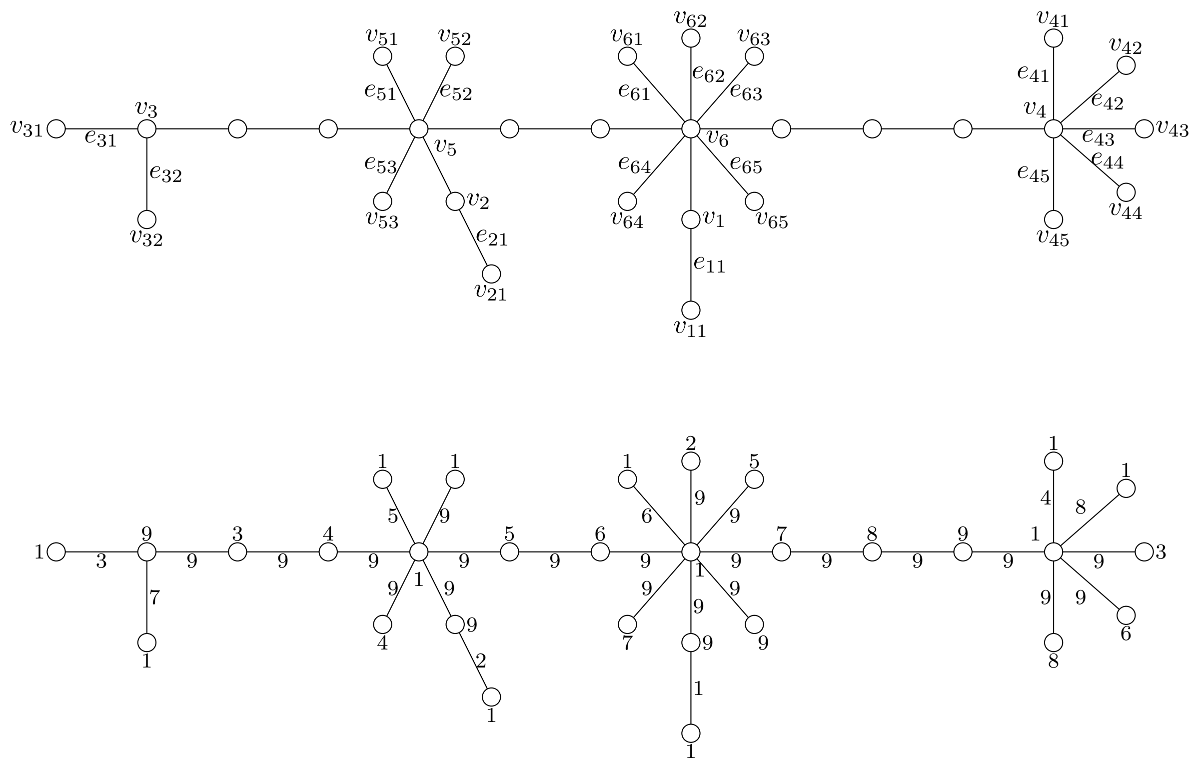

Figure 1: Example of a vertex irregular total labeling of a tree T. Top, **Step 1 and 2**: Denoting vertices in V_{Ex}, vertices and in $V_P \cup E_P$. Bottom, **Step 3, 4 and 5**: Labeling vertices and edges in $V_P \cup E$, and recursively labeling vertices in $V \backslash V_P$.

3. Conclusion

In this note, we studied the total vertex irregularity strength of trees with sufficiently large number of exterior vertices. In particular, we presented a new sufficient condition for a tree T containing n_2^e exterior vertices of degree two and containing at least $t_1 - 2n_2^e - 1$ exterior vertices of degree at least three to have $\mathrm{tvs}(T) = t_1$, which strengthens Conjecture 1. However, finding the necessary and sufficient conditions for which $\mathrm{tvs}(T) = t_1$ is still an unsolved problem. We therefore propose the following open problem.

Open Problem 1. *Find the necessary and sufficient conditions for a tree T to have* $\mathrm{tvs}(T) = t_1$.

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