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On Ramsey (mK_2, bP_n)-minimal Graphs

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Abstract

Let G and H be two given graphs. The notation $F \rightarrow (G, H)$ means that any red-blue coloring on the edges of F will create either a red subgraph G or a blue subgraph H in F. Graph F is a Ramsey (G, H)-minimal graph if F satisfies two conditions: (1) $F \rightarrow (G, H)$, and (2) $F - e \nrightarrow (G, H)$ for every $e \in E(F)$. Denote $\mathcal{R}(G, H)$ as the set of all (G, H)-minimal graphs. In this paper, we prove that a tree T is not in $\mathcal{R}(mK_2, bP_n)$ if it has a diameter of at least $n(b + m - 1) - 1$ for $m, n, b \geq 2$. Furthermore, we show that $(b + m - 1)P_n \in \mathcal{R}(mK_2, bP_n)$ for every $m, n, b \geq 2$. We also prove that for $n \geq 3$, a cycle on k vertices C_k is in $R(mK_2, bP_n)$ if and only if $k \in [n(b + m - 2) + 1, n(b + m - 1) - 1]$.

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1. Introduction

The study on Ramsey-minimal graph has received increased attention in recent years. Let F, G, and H be simple and undirected graphs. A notation $F \rightarrow (G, H)$ means that if all the edges of F are arbitrarily colored by red-blue then F will contain either a red subgraph G or a blue subgraph H. Graph F is a Ramsey (G, H)-minimal graph if $F \rightarrow (G, H)$ but $F - e \nrightarrow (G, H)$ for every $e \in E(F)$. The set of all (G, H)-minimal graphs is denoted by $\mathcal{R}(G, H)$. A red-blue coloring of edge of F such that F contains neither a red G nor a blue H is defined as a (G, H)-coloring.

The main problem of Ramsey (G, H)-minimal graph is determining graph F, which belongs to $\mathcal{R}(G, H)$ for given graphs G and H. It is also interesting to determine whether the $\mathcal{R}(G, H)$ set is finite or infinite. Burr et al. [3] showed that the set $\mathcal{R}(G, H)$ is Ramsey infinite when both G

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and H are forests, with at least one G or H having a non-star component. Burr et al. [4] proved that $\mathcal{R}(mK_2, H)$ is a Ramsey finite class for any graph H and positive integer m. They showed that the set $\mathcal{R}(K_2, H) = H$, for any graph H, $\mathcal{R}(2K_2, 2K_2) = \{3K_2, C_5\}$ and $\{2K_3, K_5, G_1\}$ are members of $\mathcal{R}(2K_2, K_3)$. The graph G_1 is given in Fig. 1.

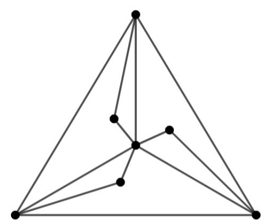

Figure 1: G_1

Burr et al. [4] also described a collection of $\frac{n+1}{2}$ non-isomorphic graphs in $\mathcal{R}(2K_2, K_n)$. Then, Mangersen and Oeckermann [5] proved that $\mathcal{R}(2K_2, K_{1,2}) = \{2K_{1,2}, C_4, C_5\}$, and presented the characterization of graphs belonging to $\mathcal{R}(2K_2, K_{1,n})$, for $n \geq 3$. Furthermore, Muhshi and Baskoro [6] proved that $\mathcal{R}(3K_2, P_3) = \{3P_3, C_4 \cup P_3, C_5 \cup P_3, C_7, C_8\}$. Baskoro and Yulianti [1] characterized all graphs in $\mathcal{R}(2K_2, P_n)$ for $n = 4, 5$. Moreover, Yulianti et al. [10] gave the construction of some infinite class in $\mathcal{R}(K_{1,2}, P_4)$. Baskoro and Wijaya [2] derived the necessary and sufficient conditions for graphs to be in $\mathcal{R}(2K_2, H)$ for any connected graph H. Wijaya and Baskoro [7] described the necessary and sufficient conditions for graphs in $\mathcal{R}(mK_2, H)$. In [8] Wijaya et al. characterized all graphs belonging to $\mathcal{R}(2K_2, K_4)$, and in [9], Wijaya et al. characterized all uncyclic graphs belonging to $\mathcal{R}(mK_2, P_3)$.

In this paper, we study the Ramsey (mK_2, bP_n)-minimal graphs for $b, m, n \geq 2$. In particular, we determine some graphs belonging to $\mathcal{R}(mK_2, bP_n)$.

2. Preliminary Results

As the starting point, the following proposition is a sufficient condition to construct a graph in $\mathcal{R}((m+1)K_2, bP_n)$ that comes from the graphs in $\mathcal{R}(mK_2, bP_n)$ for $m, n, b \geq 3$.

Proposition 2.1. *For $m, n, b \geq 3$, if $F \in \mathcal{R}(mK_2, bP_n)$ and $G \cong P_n$, then $F \cup G \in \mathcal{R}((m+1)K_2, bP_n)$.*

Proof. Let us begin with some coloring definitions. Let γ_1 be a red-blue coloring of the edge of G such that G contains a red K_2, but it has no blue P_n. We will show that $F \cup G \to ((m+1)K_2, bP_n)$. Suppose to the contrary that $F \in \mathcal{R}(mK_2, bP_n)$ and $G \cong P_n$ but $F \cup G \nrightarrow ((m+1)K_2, bP_n)$. Then there is an $((m+1)K_2, bP_n)$-coloring γ of edges of $F \cup G$, namely $\gamma(e) = \mu(e)$ for all $e \in E(F)$ and $\gamma(e) = \gamma_1(e)$ for all $e \in E(G)$. Therefore, $\mu(e)$ must be a (mK_2, bP_n)-coloring of the edge of F. This led to a contradiction with $F \to (mK_2, bP_n)$. To prove the minimality, suppose that $e \in E(F \cup G)$. It suffices to consider $e \in E(F)$. Since $F \in \mathcal{R}(mK_2, bP_n)$, then there exists an $(mK_2, bP_n)-$coloring γ_2 of the edge of $F - e$. Define a red-blue coloring ψ of edge of $F \cup G$ as follows.

$$\psi(e^*) = \begin{cases} \gamma_2(e^*), & \text{if } e^* \in E(F-e), \\ \gamma_1(e^*), & \text{otherwise.} \end{cases}$$

Therefore, we obtain an $((m+1)K_2, bP_n)$-coloring of edges of $(F \cup G) - e$. □

3. Main Results

In Theorem 3.1 we determine some graphs belonging to $\mathcal{R}(mK_2, bP_n)$ for $m, n, b \geq 2$.

Theorem 3.1. *Let $m, n, b \geq 2$. Then $(b+m-1)P_n \in \mathcal{R}(mK_2, bP_n)$.*

Proof. Denote the j^{th} path on n vertices as P_n^j for $1 \leq j \leq (b+m-1)$. The vertex set of $(b+m-1)P_n$ is $\{v_{i,j} | 1 \leq i \leq n, 1 \leq j \leq (b+m-1)\}$. Define θ_1 as the red coloring of all edges that incident to $(m-1)$ vertices, where each vertex is in a different P_n. It is obvious that $(b+m-1)P_n \rightarrow (mK_2, bP_n)$. Next, we will prove that for every $e \in E((b+m-1)P_n)$, $(b+m-1)P_n - e \nrightarrow (mK_2, bP_n)$. Without loss of the generality, let e be an edge in the j^{th} P_n, and $P_n^{\acute{j}} := P_n^j - e$. Define a red-blue coloring θ of the edge of $(b+m-1)P_n$ such that

$$\theta(e^*) = \begin{cases} \theta_1(e^*), & \text{if } e^* \in E\big((b+m-1)P_n - \{P_n^{\acute{j}}\}\big), \\ \text{blue}, & \text{otherwise.} \end{cases}$$

Then we obtain θ as the (mK_2, bP_n)-coloring of edges of $(b+m-1)P_n - e$. □

Next, in Theorem 3.2 we state the characterization of cycles that belong to $\mathcal{R}(mK_2, bP_n)$ for $m, b \geq 2$ and $n \geq 3$.

Theorem 3.2. *For $m, b \geq 2$ and $n \geq 3$, $C_k \in \mathcal{R}(mK_2, bP_n)$ if and only if $k \in [n(b+m-2)+1, n(b+m-1)-1]$.*

Proof. Define the vertex and edge sets of C_k as follows.

$$\begin{aligned} V(C_k) &= \{v_i \mid 1 \leq i \leq k\}, \\ E(C_k) &= \{e_i = v_i v_j \mid 1 \leq i \leq k,\ j = (i+1) \bmod k\}. \end{aligned}$$

First, we want to show the backward direction. Let $n(b+m-2)+1 \leq k \leq n(b+m-1)-1$. We will show that $C_k \rightarrow (mK_2, bP_n)$. Suppose that we do the maximal coloring such that there is no red mK_2. This implies that there will be at most $2(m-1)$ red edges. The red edges disconnect the graph into m^* components satisfying $m^* \leq m-1$. Let us define each component by P_{k_i}. If the rest of the edges are colored blue, then we have blue $\bigcup_{i=1}^{m^*} P_{k_i}$ with

$$\sum_{i=1}^{m^*} (k_i - 1) = |E| - 2(m-1). \tag{1}$$

Let $b_i = \lfloor \frac{k_i}{n} \rfloor$. Then b_i denotes the number of disjoint P_n contained in P_{k_i}. To prove that $C_k \to (mK_2, bP_n)$, it is sufficient to show that $\sum_{i=1}^{m^*} b_i \geq b$. According to the floor function, $k_i \leq (b_i+1)n-1$. Set all terms except k_{m^*} to the right side of equation in (1), then we have

$$k_{m^*} = |E| - 2(m-1) - \sum_{i=1}^{m^*-1} (k_i - 1) + 1. \tag{2}$$

Substituting $|E| = k$ and $k_i \leq (b_i+1)n-1$ in (2) we have

$$\begin{aligned} k_{m^*} &\geq k - 2(m-1) - \sum_{i=1}^{m^*-1} ((b_i+1)n - 1 - 1) + 1 \\ &= k - 2(m-2) - \sum_{i=1}^{m^*-1} (b_i n + n - 2) - 1 \\ &= k - 2(m-2) - \sum_{i=1}^{m^*-1} b_i n - (n-2)(m^*-1) - 1. \end{aligned} \tag{3}$$

Substituting $m^* \leq m-1$ and $k \geq n(b+m-2)+1$ in (3), we have

$$\begin{aligned} k_{m^*} &\geq n(b+m-2) + 1 - 2(m-2) - n\sum_{i=1}^{m^*-1} b_i - (n-2)(m-2) - 1 \\ &= n(b+m-2) - 2(m-2) - n\sum_{i=1}^{m^*-1} b_i - (n-2)(m-2) \\ &= n(b+m-2) - n(m-2) - n\sum_{i=1}^{m^*-1} b_i \\ &= nb - n\sum_{i=1}^{m^*-1} b_i. \end{aligned} \tag{4}$$

Hence, we compute the least value of b_{m^*} as follows.

$$b_{m^*} = \lfloor \frac{k_{m^*}}{n} \rfloor \geq b - \sum_{i=1}^{m^*-1} b_i.$$

Therefore, we have

$$\sum_{i=1}^{m^*} b_i = \sum_{i=1}^{m^*-1} b_i + b_{m^*} \geq \sum_{i=1}^{m^*-1} b_i + b - \sum_{i=1}^{m^*-1} b_i = b.$$

We conclude that $\bigcup_{i=1}^{m^*} P_{k_i}$ will contain bP_n. Then, $C_k \to (mK_2, bP_n)$.
For any $e \in E(C_k)$, $C_k - e \cong P_k$. Without loss of generality, let the path P_k has a vertex set

$V(P_k) = V(C_k)$ and an edge set $E(P_k) = E(C_k - e)$. To show that $P_k \nrightarrow (mK_2, bP_n)$, define an edge coloring $\alpha_1(e_i)$ as follows.

$$\alpha_1(e_i) = \begin{cases} \text{red}, & \text{if } i \in \{pn-1, pn\}, \\ \text{blue}, & \text{otherwise.} \end{cases}$$

for every positive $p \leq m-1$. Hence, we have blue $(\bigcup_{i=1}^{m-1} P_{n-1}) \cup P_{k^*}$ with

$$k^* = k - n(m-1). \tag{5}$$

Let $b^* = \lfloor \frac{k^*}{n} \rfloor$. Then, b^* denotes the number of disjoint P_n contained in P_{k^*}. To achieve $P_k \nrightarrow (mK_2, bP_n)$, it is sufficient to show $b^* < b$, since $\bigcup_{i=1}^{m-2} P_{n-1}$ does not contain any P_n. By assigning $k \leq n(b+m-1) - 1$ to (5), we have

$$k^* \leq n(b+m-1) - 1 - n(m-1) = nb - 1 < nb.$$

Therefore, we have

$$b^* = \lfloor \frac{k^*}{n} \rfloor < \lfloor \frac{nb}{n} \rfloor < b.$$

Hence, P_{k^*} does not contain bP_n implying $C_k - e \nrightarrow (mK_2, bP_n)$. It may be concluded that $C_k \in \mathcal{R}(mK_2, bP_n)$.

For the inverse, we need to show that $C_k \nrightarrow (mK_2, bP_n)$ if $k \leq n(b+m-2)$ and $C_k - e \rightarrow (mK_2, bP_n)$ if $k \geq n(b+m-1)$. To show that $C_k \nrightarrow (mK_2, bP_n)$, $k \leq n(b+m-2)$, define an edge coloring $\alpha_2(e_i)$ as follows

$$\alpha_2(e_i) = \begin{cases} \text{red}, & \text{if } i \in \{qn-1, qn\}, \\ \text{blue}, & \text{otherwise.} \end{cases}$$

for every non-negative $q \leq m-1$. Hence, we have blue $(\bigcup_{i=1}^{m-2} P_{n-1}) \cup P_{k^*}$ with

$$k^* = k - n(m-2) - 1. \tag{6}$$

Let $b^* = \lfloor \frac{k^*}{n} \rfloor$. Then, b^* denotes the number of disjoint P_n contained in P_{k^*}. To achieve $C_k \nrightarrow (mK_2, bP_n)$, it is sufficient to show $b^* < b$, since $\bigcup_{i=1}^{m-2} P_{n-1}$ does not contain any P_n. By assigning $k \leq n(b+m-2)$ to (6), we have

$$k^* \leq n(b+m-2) - n(m-2) - 1 = nb - 1 < nb.$$

Consequently, we have

$$b^* = \lfloor \frac{k^*}{n} \rfloor < \frac{nb}{n} = b.$$

Hence, P_{k^*} does not contain bP_n, which implies $C_k \nrightarrow (mK_2, bP_n)$.

Lastly, we need to prove $C_k - e \cong P_k \rightarrow (mK_2, bP_n)$ for $k \geq n(b+m-1)$. The proof is similar with the proof of $C_k \to (mK_2, bP_n)$, with the exception $m^* \leq m$ and $k \geq n(b+m-1)$. Starting from (3) while assigning $|E| = k-1$ and the bounds of m^* and k, we have

$$\begin{aligned} k_{m^*} &\geq |E| - 2(m-2) - \sum_{i=1}^{m^*-1} nb_i - (n-2)(m^*-1) - 1, \\ &= n(b+m-1) - 1 - 2(m-1) - n\sum_{i=1}^{m^*-1} b_i - (n-2)(m-1) + 1, \\ &= n(b+m-1) - n(m-1) - n\sum_{i=1}^{m^*-1} b_i, \\ &= nb - n\sum_{i=1}^{m^*-1} b_i. \end{aligned}$$

This result is the same as (4). Therefore, the proof may be continued similarly to have

$$\sum_{i=1}^{m^*} b_i \geq b.$$

Hence $\sum_{i=1}^{m^*} P_{k_i}$ will contain bP_n. Therefore $P_k \to (mK_2, bP_n)$. $\square$

In Theorem 3.3, we give some disconnected graph belonging to $\mathcal{R}(mK_2, (b+1)P_n)$ for $m, n \geq 3$ and $1 \leq b \leq m-2$.

Theorem 3.3. *For* $m, n \geq 3$*, if* $C_k \in \mathcal{R}(mK_2, P_n)$ *with* $k = mn-1$*, then for* $1 \leq b \leq m-2$*,* $C_k \cup bP_n \in \mathcal{R}(mK_2, (b+1)P_n)$*.*

Proof. Let us define some coloring as follows.

1. β_1 be red coloring in bP_n by coloring each edge that incident to a vertex in a different P_n,
2. β_2 be blue coloring in bP_n by coloring each edge that incident to a vertex in a different P_n,
3. β_3 be $(mK_2, P_n)-$coloring of edge of $C_k - e$ that does not contain any blue P_n.

As mentioned previously, for $k = mn-1$ and $1 \leq b \leq m-2$, we will show that $C_k \cup bP_n \to (mK_2, (b+1)P_n)$. We will divide it into two conditions. First, since $C_k \in \mathcal{R}(mK_2, P_n)$, if C_k does not have red mK_2 then C_k must have a blue P_n. In that condition, if we colored bP_n by blue, then we have $b+1$ blue P_n. Second, put the coloring β_1 on the edges of bP_n, therefore we obtain b red K_2. Then, we will show that $C_k \to ((m-b)K_2, (b+1)P_n)$. Suppose that we do not have $(m-b)$ red K_2 by taking the maximum coloring, i.e red $(m-1-b)K_2$, then from Theorem 3.2, we have that

$$\begin{aligned} &k \in [n((b+1)+(m-b)-2)+1,\ n((b+1)+(m-b)-1)-1] \\ \Leftrightarrow\ &k \in [n(m-1)+1,\ mn-1]. \end{aligned}$$

Since $k = mn - 1$ is in interval, we conclude that $C_k \to ((m-b)K_2, (b+1)P_n)$. This implies $C_k \cup bP_n \to (mK_2, (b+1)P_n)$. Other that those red-blue colorings, we always have $C_k \cup bP_n \to (mK_2, (b+1)P_n)$. Furthermore, we will show that by deleting an edge in $(C_k \cup bP_n)$, $(C_k \cup bP_n)$ has neither red mK_2 nor blue $(b+1)P_n$. It suffices to consider that edge in C_k. Since $C_k \in \mathcal{R}(mK_2, P_n)$, there is an $(mK_2, P_n)-$coloring of edge of $C_k - e$. Define a red-blue coloring β on the edge of $C_k \cup bP_n$ as follows.

$$\beta(e^*) = \begin{cases} \beta_3(e^*) & \text{if } e^* \in E(C_k - e), \\ \beta_2(e^*) & \text{otherwise.} \end{cases}$$

Therefore we obtain an $(mK_2, (b+1)P_n)$-coloring of edges of $(C_k \cup bP_n) - e$. □

In the following theorem, we prove that a tree with certain diameter is not in $\mathcal{R}(mK_2, bP_n)$ for $m, n, b \geq 2$.

Theorem 3.4. *Let* $m, n, b \geq 2$*. If a tree* T *has a diameter at least* $n(b+m-1)-1$*, then* $T \notin \mathcal{R}(mK_2, bP_n)$*.*

Proof. Let T be a tree with $diam(T) \geq n(b+m-1)-1$. Suppose that there is a $T \in \mathcal{R}(mK_2, bP_n)$. Let L be the longest path between vertex u_L and v_L in T which has a distance $d(u_L, v_L) \geq n(b+m-1)-1$, then choose an edge $e \in E(L)$ such that removing e makes L breaks into two components, $G_1 \cong P_n$ and $G_2 \cong L - G_1 - e$. Then, observe that $diam(G_2) \geq n(b+m-1)-1-n$ or G_2 has at least $(n(b+m-1)-n)$ vertices. From the proof of Theorem 3.2, we obtain $G_2 \to (mK_2, (b-1)P_n)$. Let ϕ_1 be a red-blue coloring of the edge of G_1 such that G_1 contains a blue P_n but no red K_2. Next, let ϕ_2 be a red-blue coloring of the edge of G_2 such that $G_2 \to (mK_2, (b-1)P_n)$. Define a red-blue coloring ϕ as follows.

$$\phi(e) = \begin{cases} \phi_1(e), & \text{if } e \in E(G_1), \\ \phi_2(e), & \text{if } e \in E(G_2). \end{cases}$$

We obtain either red mK_2 or blue bP_n in $L - e$, a contradiction to the minimality. □

4. Conclusion

In this paper we prove that for $m, n, b \geq 2$, $(b+m-1)P_n \in \mathcal{R}(mK_2, bP_n)$. We also give the characterization of a cycle C_k to be in the class $\mathcal{R}(mK_2, bP_n)$ for $m, b \geq 2$ and $n \geq 3$. Next, we show that for $m, n \geq 3$, $C_k \cup bP_n \in \mathcal{R}(mK_2, (b+1)P_n)$, provided that $C_k \in \mathcal{R}(mK_2, bP_n)$ for $1 \leq b \leq m-2$. We also state that a tree T is not in $\mathcal{R}(mK_2, bP_n)$ if it has a diameter at least $n(b+m-1)-1$ for $m, n, b \geq 2$.

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