Some results on cordiality labeling of generalized Jahangir graph

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Abstract

In this paper we consider the cordiality of a generalized Jahangir graph \(J_{n,m}\). We give sufficient condition for \(J_{n,m}\) to admit (or not admit) the prime cordial labeling, product cordial labeling and total product cordial labeling.

Keywords: Generalized Jahangir graph, Prime cordial labeling, Product cordial labeling, Total product cordial labeling

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1. Introduction

Let \(G = (V, E)\) be the connected, simple and undirected graph with vertex set \(V\) and edge set \(E(G)\). For standard terminology and notations in Graph Theory, we refer [18]. By a labeling we mean any mapping that carries a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain of the mapping is the set of vertices or the set of edges, then the labeling is called a vertex labeling or (edge labeling). If the domain is \(V \cup E\) then we call the labeling as total labeling. Many labeling schemes have been introduced so far and they are well explored by many researchers. For a dynamic survey on various graph labeling problems, we refer to Gallian [3].

A labeling \(f : V(G) \rightarrow \{0, 1\}\) is called binary vertex labeling of \(G\) and \(f(v)\) is called the label of the vertex \(v\) of \(G\) under \(f\). If for an edge \(e = uv\), the induced edge labeling \(f^* : E(G) \rightarrow \{0, 1\}\)
is given by $f^*(e) = |f(u) - f(v)|$. Then $v_f(i)$ is the number of vertices of $G$ having label $i$ under $f$ and $e_f(i)$ is the number of edges of $G$ having label $i$ under $f$, where $i = 0$ or $1$.

Definition 1. A binary vertex labeling $f$ of a graph $G$ is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [2] as a weaker version of graceful labeling and harmonious labeling.

The notion of prime labeling was originated by Entringer and was introduced by Tout et al. [17]. Motivated by the concepts of prime labeling and cordial labeling, a new concept termed as a product cordial labeling function $f$ is given by

$f^*(e = uv) = \begin{cases} 
1, & \text{if } \gcd(f(u), f(v)) = 1 \\
0, & \text{if } \gcd(f(u), f(v)) > 1 
\end{cases}$

satisfies the condition $|e_f(0) - e_f(1)| \leq 1$. The graph admits a prime cordial labeling is called a prime cordial graph.

Many graph families proved to be prime cordial, for example see [8, 12, 13, 14].

In 2004, Sundaram et al. [9] introduced the product cordial labeling of graph.

Definition 3. Let $f : V(G) \rightarrow \{0, 1\}$ be a vertex labeling of a graph $G$ that induces an edge labeling function $f^* : E(G) \rightarrow \{0, 1\}$ such that $f^*(uv) = f(u)f(v)$ where $uv \in E(G)$. Then $f$ is a product cordial labeling if $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. A graph $G$ is product cordial if it admits a product cordial labeling.

In 2006, Sundaram et al. [10] introduced the notion of total product cordial labeling of graph.

Definition 4. Let $f : V(G) \rightarrow \{0, 1\}$ be a vertex labeling of a graph $G$ that induces an edge labeling function $f^* : E(G) \rightarrow \{0, 1\}$ such that $f^*(uv) = f(u)f(v)$ where $uv \in E(G)$. Then $f$ is a total product cordial labeling if $|v_f(1) + e_f(1)) - (v_f(0) + e_f(0))| \leq 1$. A graph $G$ is total product cordial if it admits a total product cordial labeling.

For more results on product cordial and total product cordial, please refer [4, 5, 6, 9, 10, 11].

For $n, m \geq 2$, the generalized Jahangir graph $J_{n,m}$ is a graph on $nm + 1$ vertices, that is, the graph consists of a cycle $C_{mn}$ with one additional vertex which adjacent to a $m$ vertices of $C_{mn}$ at distance $n$ to each other on $C_{mn}$, see [1, 7]. The following figure shows the graph $J_{n,m}$ for $n = 3$ and $m = 10$.

In this paper, we investigate prime cordial labeling, product cordial labeling and total product cordial labeling of generalized Jahangir graph $J_{n,m}$.

2. Main Results

In this section, we present our main results.
Lemma 2.1. The Jahangir graph $J_{2,m}$, $m \geq 4$ is prime cordial.

Proof. Let $J_{2,m}$, $m \geq 4$ be Jahangir graph with the vertex set $V(J_{2,m}) = \{v\} \cup \{v_i : 1 \leq i \leq 2m\}$ and the edge set $E(J_{2,m}) = \{v_iv_{i+1} : 1 \leq i \leq 2m-1\} \cup \{v_1v_1\} \cup \{vv_{2i-1} : 1 \leq i \leq m\}$. Clearly that $|V(J_{2,m})| = 2m + 1$ and $|E(J_{2,m})| = 3m$.

To show that $J_{2,m}$ is a prime cordial, we define a vertex labeling $f : V(J_{2,m}) \rightarrow \{1, 2, \ldots, 2m+1\}$ in the following way:

$$f(v) = 2, f(v_1) = 6, f(v_2) = 4, f(v_{2m}) = 3, f(v_{2m-1}) = 9, f(v_{2m-2}) = 5, f(v_{2m-3}) = 7, f(v_{2m-4}) = 1$$

$$f(v_i) = \begin{cases} 2(i + 1), & \text{if } 3 \leq i \leq \lfloor \frac{2m-1}{2} \rfloor \\ 4m - 2i + 1, & \text{if } \lfloor \frac{2m-1}{2} \rfloor + 1 \leq i \leq 2m - 5 \end{cases}$$

We have $|e_f(0)| = \lfloor \frac{3m}{2} \rfloor$ and $|e_f(1)| = \lceil \frac{3m}{2} \rceil$. Then $|e_f(0) - e_f(1)| \leq 1$. Hence, the Jahangir graph $J_{2,m}$ is prime cordial.

This completes the proof. \qed

Theorem 2.1. The Jahangir graph $J_{n,m}$, $n > 2$, $m > 3$ is prime cordial.

Proof. Let $J_{n,m}$, $n > 2$, $m > 3$ be Jahangir graph with the vertex set $V(J_{n,m}) = \{v\} \cup \{v_i : 1 \leq i \leq mn\}$ and the edge set $E(J_{n,m}) = \{v_iv_{i+1} : 1 \leq i \leq mn-1\} \cup \{v_mv_1\} \cup \{vv_{n(i-1)+1} : 1 \leq i \leq m\}$. Clearly that $|V(J_{n,m})| = mn + 1$ and $|E(J_{n,m})| = n(m + 1)$.

To show that $J_{n,m}$ is prime cordial, we define vertex labeling $f : V(J_{n,m}) \rightarrow \{1, 2, \ldots, mn+1\}$ as follows:

$$f(v) = 2, f(v_1) = 6, f(v_2) = 4, f(v_m) = 3$$

$$f(v_{mn-1}) = \begin{cases} 1, & \text{for } m, n \text{ are odd} \\ 9, & \text{for otherwise} \end{cases}$$
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\[
f(v_{mn-2}) = \begin{cases} 
5, & \text{for } m, n \text{ are even} \\
 & \text{or } m, n \text{ are odd} \\
1, & \text{for otherwise}
\end{cases}
\]

\[
f(v_{mn-3}) = \begin{cases} 
7, & \text{if } m, n \text{ are even} \\
 & \text{or } m, n \text{ are odd} \\
5, & \text{for otherwise}
\end{cases}
\]

\[
f(v_{mn-4}) = \begin{cases} 
1, & \text{if } m, n \text{ are even} \\
9, & \text{if } m, n \text{ are odd} \\
7, & \text{for otherwise}
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
2(i + 1), & \text{if } 3 \leq i \leq \left\lfloor \frac{mn-1}{2} \right\rfloor \\
2mn - 2i + 1, & \text{if } \left\lfloor \frac{mn-1}{2} \right\rfloor + 1 \leq i \leq mn - 5.
\end{cases}
\]

We have

\[
|e_f(0)| = \begin{cases} 
\frac{(m(n+1)-1)}{2}, & \text{if } m \text{ are even} \\
 & \text{and } n \text{ are odd} \\
\frac{(m(n+1))}{2}, & \text{for otherwise}
\end{cases}
\]

\[
|e_f(1)| = \begin{cases} 
\frac{(m(n+1)+1)}{2}, & \text{if } m \text{ are even} \\
 & \text{and } n \text{ are odd} \\
\frac{(m(n+1))}{2}, & \text{for otherwise}
\end{cases}
\]

It is easy to show that \( |e_f(0) - e_f(1)| \leq 1 \). Hence, the Jahangir graph \( J_{n,m} \) is prime cordial. This completes the proof.

The following figure illustrates the prime cordial labeling of graph \( J_{3,5} \).

**Theorem 2.2.** The Jahangir graph \( J_{n,m} \) is product cordial with \( n \geq 2, m \geq 3, m \text{ is odd and } n \text{ is even.} \)

**Proof.** Let \( J_{n,m} \) with \( n \) is even and \( n \geq 2, m \) is odd and \( m \geq 3 \), be Jahangir graph with the vertex set \( V(J_{n,m}) = \{v\} \cup \{v_i : 1 \leq i \leq mn\} \) and the edge set \( E(J_{n,m}) = \{v_iv_{i+1} : 1 \leq i \leq mn - 1\} \cup \{v_{mn}v_1\} \cup \{v_{n(i-1)+1} : 1 \leq i \leq m\} \). Clearly that \( |V(J_{n,m})| = mn + 1 \) and \( |E(J_{n,m})| = m(n + 1) \).
To show that $J_{n,m}$ is product cordial, define a vertex labeling $f : V(J_{n,m}) \rightarrow \{0, 1\}$ in the following way:

\[
f(v) = 1
\]

\[
f(v_i) = \begin{cases} 
1, & \text{if } 1 \leq i \leq \frac{nm}{2} \\
0, & \text{if } \frac{nm}{2} + 1 \leq i \leq mn
\end{cases}
\]

From the above labeling, we can see that $v_f(1) = \frac{mn+2}{2}$, $v_f(0) = \frac{mn}{2}$, $e_f(1) = \frac{m(n+1)-1}{2}$, $e_f(0) = \frac{m(n+1)+1}{2}$. Hence $|v_f(1) - v_f(0)| = 1$ and $|e_f(1) - e_f(0)| = 1$. Therefore the graph $J_{n,m}$ is product cordial.

This completes the proof.

Figure 3 below illustrates the product cordial labeling of graph $J_{2,5}$.

In Theorem 2.2, the graph $J_{n,m}$ is product cordial labeling for $n \geq 2$, $m \geq 3$, $m$ is odd and $n$ is even. We have tried to find the product cordial labeling of $J_{n,m}$ for all values of $m$ and $n$ but so far without success. So we pose the following open problem.

**Problem 1.** Determine product cordial labeling of the Jahangir graph $J_{n,m}$ for all $m$ and $n$. 

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Theorem 2.3. The Jahangir graph $J_{n,m}$, $n \geq 2$, $m \geq 3$ is total product cordial.

Proof. Let $J_{n,m}$, $n \geq 2$, $m \geq 3$ be a Jahangir graph with the vertex set $V(J_{n,m}) = \{v\} \cup \{v_i : 1 \leq i \leq mn\}$ and the edge set $E(J_{n,m}) = \{v_iv_{i+1} : 1 \leq i \leq mn - 1\} \cup \{v_{mn}v_1\} \cup \{v_{v_{(i-1)+1}} : 1 \leq i \leq m\}$. Clearly that $|V(J_{n,m})| = mn + 1$ and $|E(J_{n,m})| = m(n + 1)$.

To show that $J_{n,m}$ is total product cordial, define a vertex labeling $f : V(J_{n,m}) \rightarrow \{0, 1\}$ in the following way:

**Case 1:** $m$ and $n$ are odd.

\[ f(v) = 1, f(v_{mn-1}) = 1 \]

\[ f(v_i) = \begin{cases} 
1, & \text{if } 1 \leq i \leq \frac{mn - 1}{2} \\
0, & \text{if } \frac{mn + 1}{2} \leq i \leq mn, \ i \neq mn - 1
\end{cases} \]

We have $|v_f(1)| = \lceil mn + 2 \rceil$, $|v_f(0)| = \frac{mn}{2}$, $|e_f(1)| = \frac{m(n+1)-2}{2}$, $|e_f(0)| = \frac{m(n+1)+2}{2}$. It is easy to see that \(|(v_f(1) + e_f(1)) - (v_f(0) + e_f(0))| \leq 1\). Hence the graph $J_{n,m}$ is total product cordial.

**Case 2:** $m$ and $n$ are not odd.

\[ f(v) = 1 \]

\[ f(v_i) = \begin{cases} 
1, & \text{if } 1 \leq i \leq \frac{mn}{2} \\
0, & \text{if } \frac{mn + 2}{2} \leq i \leq mn
\end{cases} \]

We have $|v_f(1)| = \frac{mn + 2}{2}$, $|v_f(0)| = \frac{mn}{2}$, $|e_f(1)| = \frac{m(n+1)-1}{2}$, $|e_f(0)| = \frac{m(n+1)+1}{2}$. It is easy to see that \(|(v_f(1) + e_f(1)) - (v_f(0) + e_f(0))| \leq 1\). Hence the graph $J_{n,m}$ is total product cordial.

This completes the proof.

Figure 4 shows the total product cordial labeling of graph $J_{4,5}$.
In [15, 16], Vaidya and Barasara introduced an edge product cordial labeling and a total edge product cordial labeling of graph $G$. Thus, we propose the following problem.

**Problem 2.** Determine edge product cordial labeling and total edge product cordial labeling of the Jahangir graph $J_{n,m}$ for $n, m \geq 2$.

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