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All unicyclic graphs of order n with locating-chromatic number $n-3$

Edy Tri Baskoro[a], Arfin[b]

[a]*Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesa 10, Bandung 40132, Indonesia.*
[b]*Master Program in Mathematics for Teaching, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesa 10, Bandung 40132, Indonesia*

ebaskoro@math.itb.ac.id, yong_arfin@yahoo.co.id

Abstract

Characterizing all graphs having a certain locating-chromatic number is not an easy task. In this paper, we are going to pay attention on finding all unicyclic graphs of order n (≥ 6) and having locating-chromatic number $n-3$.

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1. Introduction

Let $G = (V, E)$ be a connected graph. For any two vertices a and b in G, define the *distance* between a and b, denoted by $d(a, b)$, is the length of a shortest path connecting a and b. The *distance* from a vertex a to a set S in G, denoted by $d(a, S)$, is $\min\{d(a, x) \mid x \in S\}$. Let $\Pi = \{L_1, L_2, ..., L_k\}$ be an ordered partition of $V(G)$ induced by a k-coloring c. The *color code* $c_\Pi(v)$ of a vertex v of G is defined as

$$c_\Pi(v) = (d(v, L_1), d(v, L_2), ..., d(v, L_k)).$$

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If any two distinct vertices u and v of G satisfy that $c_\Pi(u) \neq c_\Pi(v)$, then the coloring c is called a *locating-coloring* of G. The *locating-chromatic number* of G, denoted by $\chi_L(G)$, is the smallest integer k such that G has a locating-coloring with k colors.

Chartrand et al. [5] introduced the notion of the locating-chromatic number of a graph. They derived some bounds of the locating-chromatic number of a graph in terms of its order and diameter. The locating-chromatic numbers of some well-known graphs are also obtained, such as for paths, cycles, double stars, and complete multipartite graphs. The existence of a tree of order $n \geq 5$ having locating-chromatic number k for any $k \in \{3, 4, ..., n-2, n\}$ is also shown. In [8], Furuya and Matsumoto have proposed an algorithm to estimate an upper bound for the locating-chromatic number of any tree. This bound depends on the number of leaves and the number of local end-branches in a tree. Recently, Assiyatun et al. [3] proposed an improved algorithm for calculating the upper bound for the locating-chromatic number of any tree. The bound obtained is much better than the one of Furuya and Matsumoto.

All connected graphs of order n and having locating-chromatic number n have been completely characterised, i.e., complete multipartite graphs, see [5]. For small locating-chromatic number, all connected graphs with locating-chromatic number 3 have been characterized, see [4] and [2]. In particular for trees, Syofyan et al. [9] has found all trees of order n with locating-chromatic number t, where $2 \leq t < \frac{n}{2}$. Furthermore, in [6], Chartrand et al. characterized all connected graphs of order n and having locating-chromatic number $n-1$. However, the problem on characterizing all connected graphs of order n and having locating-chromatic number $n-2$ is still open. A graph is called *unicyclic* if it contains exactly one cycle. Recently, Arfin and Baskoro [1] characterized all unicyclic graphs of order $n \geq 5$ with locating-chromatic number $n-2$. Such graphs are presented in the following theorem. In this paper, we characterize all unicyclic graphs of order n (≥ 6) with locating-chromatic number $n-3$.

Theorem 1.1. *[1] There are exactly 9 non-isomorphic unicyclic graphs of order $n \geq 5$, listed in Figure 1, with locating-chromatic number $n-2$.*

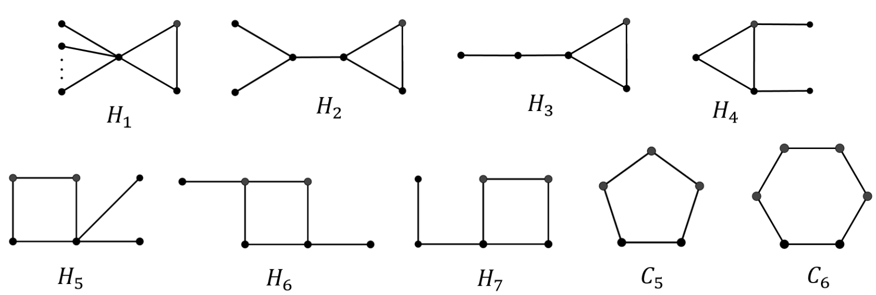

Figure 1. All unicyclic graphs of order $n \geq 5$ with locating-chromatic number $n-2$.

2. Basic Properties

In this section, we give some basic properties of locating-chromatic number of graphs. Let $G(V, E)$ be a nonempty connected graph of order n. The *degree* of vertex v in G, denoted by

$deg(v)$, is the number of vertices in G that are adjacent to v. A vertex of degree one is called an *end-vertex* or a *leaf* of G. The *external degree* of a vertex v in G, denoted by $d^+(v)$, is the number of leaves adjacent to v. The *maximum external degree* of a graph G is $\max\{d^+(v) \mid v \in V(G)\}$ and denoted by $\Delta^+(G)$. The set of all vertices adjacent to vertex v in G is denoted by $N(v)$. The following observation and corollary are natural.

Observation 2.1. *[5] Let c be a locating-coloring in a connected graph G. If u and v are distinct vertices of G such that $d(u,w) = d(v,w)$ for all $w \in V(G)\backslash\{u,v\}$, then $c(u) \neq c(v)$. In particular, if u and v are nonadjacent vertices of G such that $N(u) = N(v)$, then $c(u) \neq c(v)$.*

Corollary 2.1. *[5] If G is a connected graph containing a vertex v with $d^+(v) = p$, then $\chi_L(G) \geq p+1$. Furthermore, if $\Delta^+(G) = P$, then $\chi_L(G) \geq P+1$.*

Furthermore, Chartrand, et al. [5] derived some bounds on the locating-chromatic number of a connected graph in relation with its order and diameter, as shown in the following theorem.

Theorem 2.1. *[5] If G is a graph of order $n \geq 3$ and $diam(G) \geq 2$, then*

$$\log_{d+1} n \leq \chi_L(G) \leq n - diam(G) + 2$$

Note that $diam(G)$ is the diameter of graph G. As a direct consequence of Theorem 2.1, we have the following corollaries.

Corollary 2.2. *If G is a graph of order $n \geq 6$ with locating-chromatic number $n-3$, then $2 \leq diam(G) \leq 5$.*

Corollary 2.3. *If k is the length of a cycle in a unicyclic graph G of order n (≥ 6) with locating-chromatic number $n-3$, then $3 \leq k \leq 11$.*

A tree T for which a vertex v is distinguished is called a *rooted tree* and the distinguished vertex is called a *root* of the tree. A rooted tree will be considered to be *leveled*, *i.e.* level 0 contains the root, v, of the tree, level 1 consists of all vertices adjacent to v, etc. A rooted tree T is called *trivial* if it is of order 1, otherwise it is *nontrivial*. Let H be a unicyclic graph containing a cycle of length k. Then, the graph H can be also considered as the graph obtained from k rooted trees T_i of roots $a_i (1 \leq i \leq k)$ by connecting all these roots into a cycle C_k such that:

$$V(H) = \bigcup_{i=1}^{k} V(T_i) \text{ and } E(H) = \left(\bigcup_{i=1}^{k} E(T_i)\right) \cup E(C_k).$$

In this paper, we denote by $\mathbb{H}$ the set of all unicyclic graphs H of order $n \geq 6$ with $\chi_L(H) = n-3$. Note that there is no such unicyclic graph H of order $n \leq 5$ with $\chi_L(H) = n-3$.

3. Maximum external degree

In this section, we are going to show that every unicyclic graph H of order $n \geq 8$ with $\chi_L(H) = n-3$ must have the maximum external degree $n-4$, namely $\Delta^+(H) = n-4$. To do this, let us first consider the following lemma.

Lemma 3.1. *If H is a unicyclic graph of order $n \geq 8$ with $\Delta^+(H) = 1$, then $\chi_L(H) \leq n-4$.*

Proof. Let H be a unicyclic graph of order $n \geq 8$ with $\Delta^+(H) = 1$. Let k be the length of the unique cycle in H. Then, consider the following two cases.

Case 1: $3 \leq k \leq 7$. Consider any connected subgraph I of H of order 8 and containing the unique cycle with $\Delta^+(I) = 1$. Then, all these possible subgraphs I for each k are shown in Figures 2 and 3, along with their minimum locating-colorings.

It can be seen that every subgraph I in Figures 2 and 3 has a minimum locating-coloring with either 3 or 4 colors. Now extend this coloring into H by coloring all the remaining vertices in H with new different colors. By this way, we obtain a locating-coloring in H with at most $(n-8)+4 = n-4$ colors. Therefore, $\chi_L(H) \leq n-4$.

Case 2: $k \geq 8$. Now, consider the unique cycle C_k in H and let $V(C_k) = \{a_i \mid 1 \leq i \leq k\}$. If k is odd, then define a coloring $c : V(C_k) \to \{1,2,3\}$ with:

$$c(a_i) = \begin{cases} 1, & \text{if } i = 1 \\ 2, & \text{if } i \text{ is even} \\ 3, & \text{if } i \geq 3 \text{ and } i \text{ is odd.} \end{cases}$$

If k is even, then define a coloring $c : V(C_k) \to \{1,2,3,4\}$ with:

$$c(a_i) = \begin{cases} 1, & \text{if } i = 1 \\ 2, & \text{if } i = 2 \\ 3, & \text{if } i \geq 3 \text{ and } i \text{ is odd} \\ 4, & \text{if } i \geq 4 \text{ and } i \text{ is even.} \end{cases}$$

It is clear that c is a locating-coloring in C_k. Now, extend this coloring c into H by coloring all the remaining vertices in H with new different colors. Of course, this extended coloring is a locating-coloring in H. Then, we obtain $\chi_L(H) \leq n-4$. □

Theorem 3.1. *If H is a unicyclic graph of order $n \geq 8$ with $\chi_L(H) = n-3$, then $\Delta^+(H) = n-4$.*

Proof. Let H be a unicyclic graph of order $n \geq 8$ with $\chi_L(H) = n-3$. If $\Delta^+(H) \geq n-3$, then by Corollary 2.1 we have $\chi_L(H) \geq n-3+1 = n-2$, a contradiction. Therefore, $\Delta^+(H) \leq n-4$. Now, assume $\Delta^+(H) < n-4$. Let x be a vertex with maximum external degree, *i.e.* $d^+(x) = \Delta^+(H) \leq n-5$.

If $\Delta^+(H) = 0$, it follows that $H \cong C_n$ which means $\chi_L(H) = 3$ for odd n or 4 for even n, a contradiction. If $\Delta^+(H) = 1$, then by Lemma 3.1, we have $\chi_L(H) \leq n-4$, a contradiction. Therefore, $2 \leq \Delta^+(H) \leq n-5$. Let $u_1, u_2, \cdots, u_{\Delta^+(H)}$ be the leaves adjacent to x in H. By Corollary 2.1 the vertices $x, u_1, u_2, \cdots, u_{\Delta^+(H)}$ must be assigned with distinct colors, say $1, 2, \cdots, \Delta^+(H)+1$. Now, consider the remaining vertices other than x and its leaves in H.

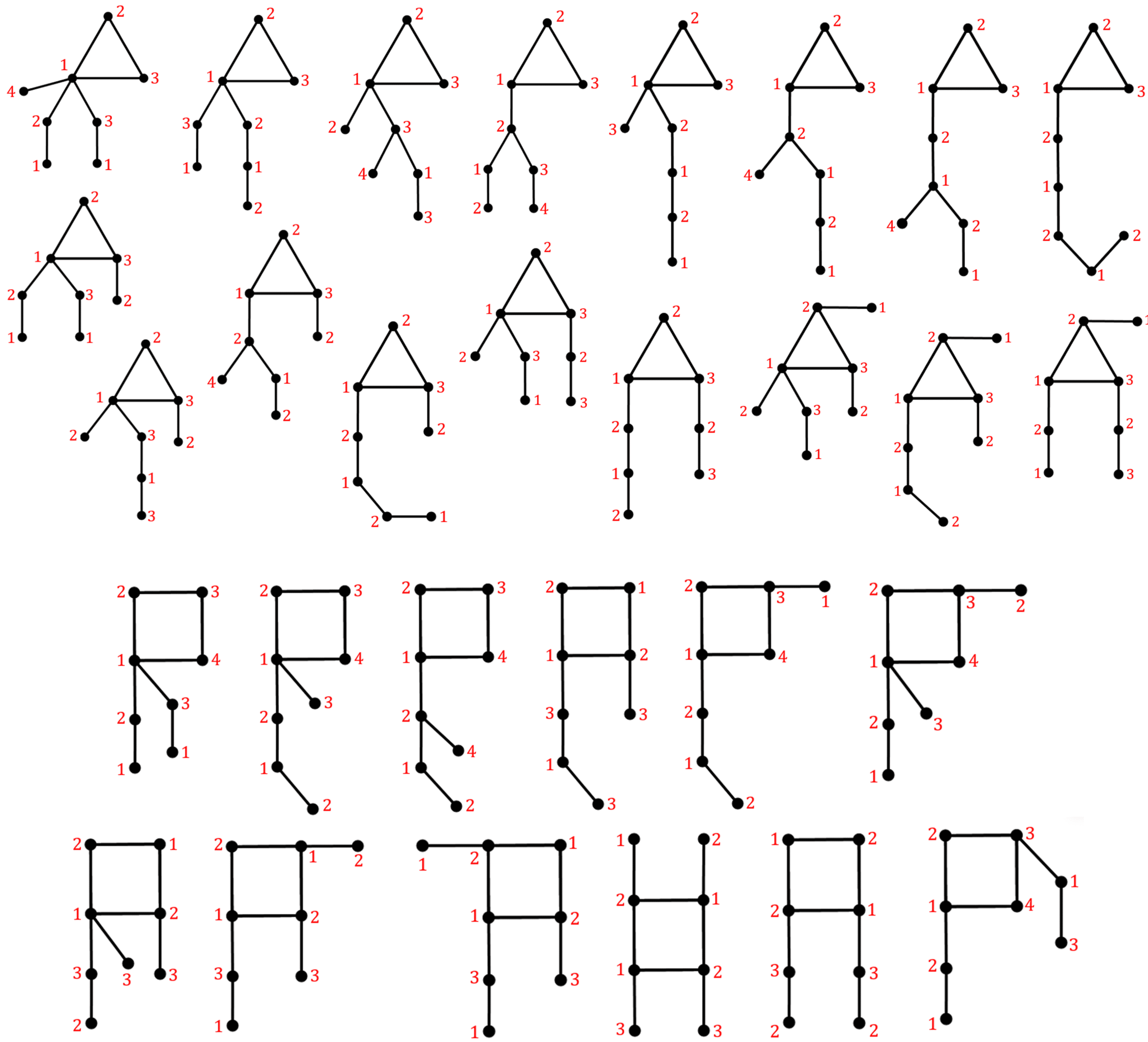

Figure 2. All the subgraphs I of $k = 3$ or $k = 4$ along with their minimum locating-colorings.

Let J be a subgraph induced by these remaining vertices, say $V(J) = \{v_1, v_2, \cdots, v_{n-\Delta^+(H)-1}\}$. Then, there are at least 5 vertices in J. Let p and q be two non-adjacent vertices in J such that $d(p,w) \neq d(q,w)$ for some $w \in V(H)\backslash\{p,q\}$. Define a coloring such that p and q are assigned with the same color, and the other $n - \Delta^+(H) - 3$ remaining vertices in J are assigned with distinct colors different from the colors of p and q. Such a coloring of H is a locating-coloring, hence $\chi_L(H) \leq \max\{\Delta^+(H)+1, n-\Delta^+(H)-2\} \leq n-4$, which is a contradiction. Therefore, $\Delta^+(H) = n-4$. □

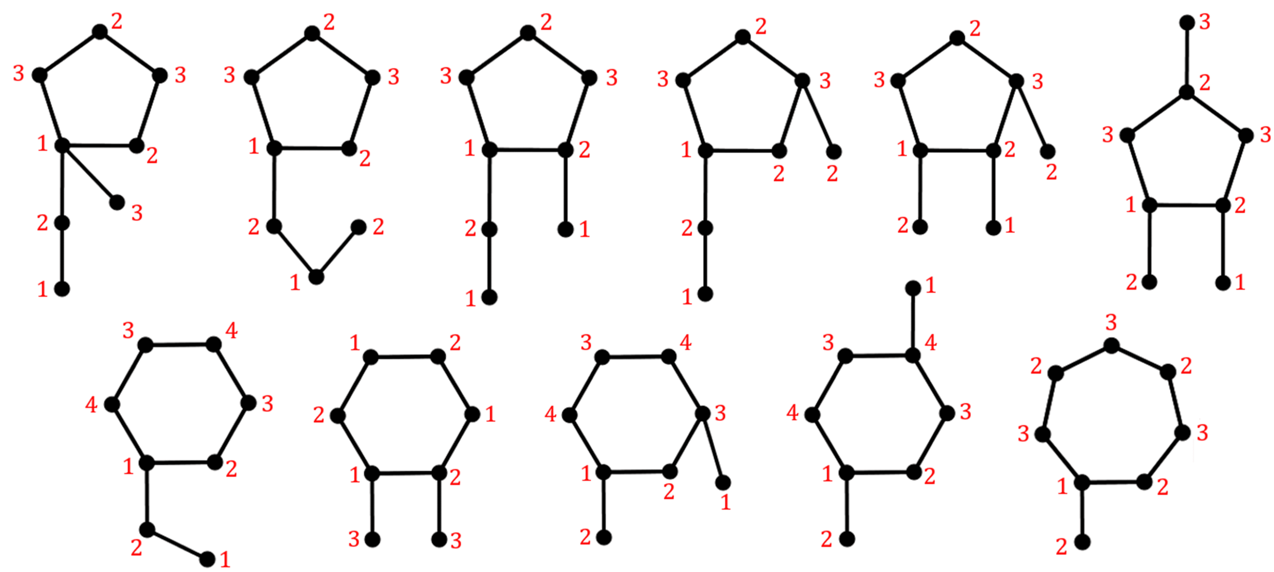

Figure 3. All the subgraphs I with $k = 5, 6$, or $k = 7$ along with their minimum locating-colorings.

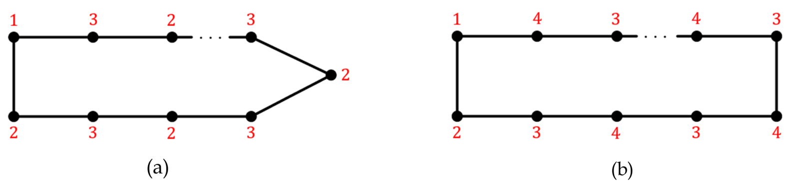

Figure 4. A locating-coloring c in the unique cycle.

4. Characterization

Let H be a unicyclic graph of order $n \geq 6$ with $\chi_L(H) = n - 3$. In this section, we will characterize all graphs H.

Theorem 4.1. *There are exactly three non-isomorphic unicyclic graphs H of order $n \geq 8$ with $\chi_L(H) = n - 3$.*

Proof. Let H be a unicyclic graph of order $n \geq 8$ and $\chi_L(H) = n - 3$. By Theorem 3.1, we have $\Delta^+(H) = n - 4$. Let x be a vertex of H with maximum external degree, *i.e.* $d^+(x) = \Delta^+(H) = n - 4$. Then, there are exactly three remaining vertices other than x and its leaves. The connected subgraph induced by these three vertices together with x will contain a unique cycle. Therefore, there are exactly three possible graphs H up to isomorphism (see Figure 5). For the converse, by Corollary 2.1, we have that $\chi_L(H) \geq n - 3$. Next, each of these three graphs has a locating-coloring with $n - 3$ colors (see Figure 5), hence $\chi_L(H) \leq n - 3$. Therefore, for each of these graphs H, we have $\chi_L(H) = n - 3$. □

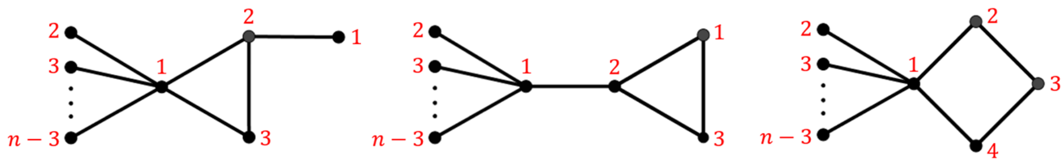

Figure 5. Three non-isomorphic unicyclic graphs H of order n and $\chi_L(H) = n-3$ with their minimum locating-colorings.

To complete the characterization, we have to find all the unicyclic graphs H of order $n \leq 7$ with the required locating-chromatic number. Our search will be based on the length of the unique cycle C_k in H.

Theorem 4.2. *There are exactly two non-isomorphic unicyclic graphs H of order $n \leq 7$ with $\chi_L(H) = n-3$ containing C_k for $k \geq 5$.*

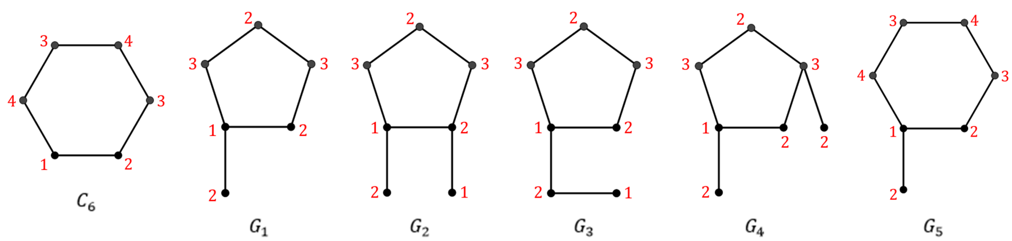

Figure 6. Graphs C_6, G_1, G_2, G_3, G_4, and G_5, each with its minimum locating-coloring.

Proof. Let H be a unicyclic graph of order $n \leq 7$ with $\chi_L(H) = n-3$ and containing the cycle of length $k \geq 5$. Then, $k = 5, 6$, or 7. If $k = 7$ then $H \cong C_7$ and $\chi_L(C_7) = 3$ $(= n-4)$, a contradiction. If $k = 5$ or 6, then H must be isomorphic to C_6, G_1, G_2, G_3, G_4, or G_5 (see Figure 6). We can see that G_1 and G_5 are the only graphs having the required locating-chromatic number. □

Theorem 4.3. *There are exactly 12 non-isomorphic unicyclic graphs H of order $n \leq 7$ containing C_3 with $\chi_L(H) = n-3$.*

Proof. Let H be a unicyclic graph of order $n \leq 7$ containing C_3. Since the order of H must be at least 6, then H must be a connected graph obtained from three rooted trees of total order $n = 6$ or $n = 7$, by connecting all roots into such a cycle C_3. By Corollary 2.2, the diameter of H is at least 2 and at most 5. These restrictions lead to 25 possible graphs H up to isomorphism, as shown in Figure 7 with their minimum locating-colorings. Thus, there are only 12 of them having the required locating-chromatic number (inside the blue square). □

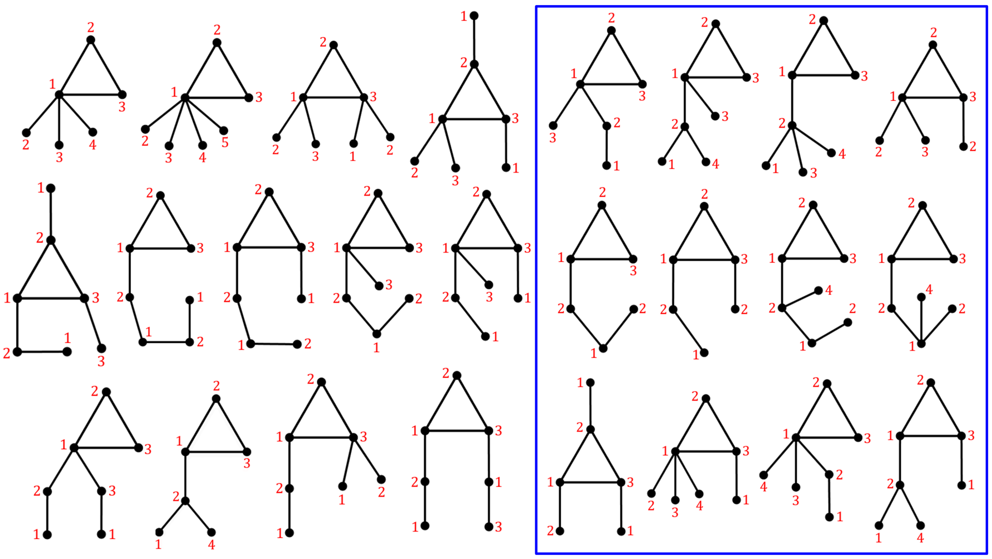

Figure 7. All possible graphs H of order $n \leq 7$ containing C_3 with their minimum locating-colorings.

Theorem 4.4. *There are exactly 8 non-isomorphic unicyclic graphs H of order $n \leq 7$ containing C_4 with $\chi_L(H) = n - 3$.*

Proof. Let H be a unicyclic graph of order $n \leq 7$ containing C_4. Since the order of H must be at least 6, then H must be a connected graph obtained from three rooted trees of total order $n = 6$ or $n = 7$, by connecting all roots into such a cycle C_4. By Corollary 2.2, the diameter of H is at least 2 and at most 5. These restrictions lead to 13 possible graphs H up to isomorphism, as shown in Figure 8 with their minimum locating-colorings. Thus, there are only 8 of them having the required locating-chromatic number (inside the blue square), hence it completes the proof. □

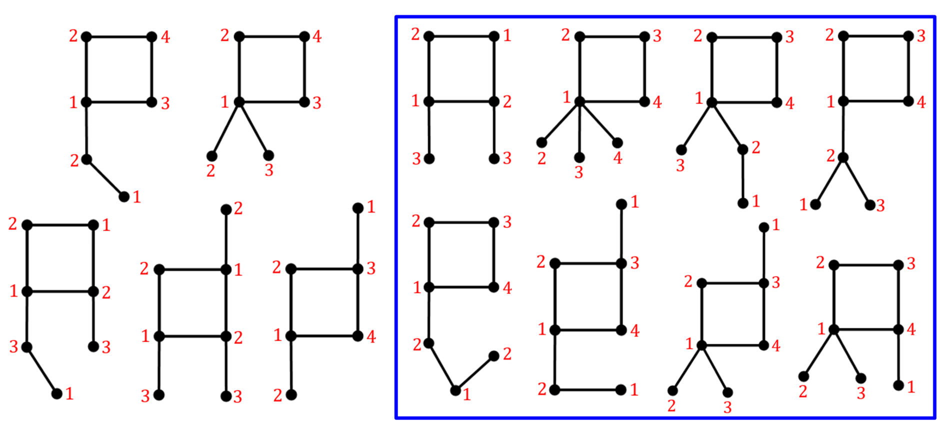

Figure 8. All possible graphs H of order $n \leq 7$ containing C_4 with their minimum locating-colorings.

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